三角变换公式

两角和公式：$\sin(A+B)=\sin A\cos B+\cos A\sin B$　$\sin(A-B)=\sin A\cos B-\cos A\sin B$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$　$\cos(A-B)=\cos A\cos B+\sin A\sin B$

$\tan(A+B)=(\tan A+\tan B)/(1-\tan A\tan B)$

$\tan(A-B)=(\tan A-\tan B)/(1+\tan A\tan B)$

倍角公式：$\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$

$\sin 2A=2\sin A\cos A$

$\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$

和差化积：$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$2\cos A\sin B=\sin(A+B)-\sin(A-B)$

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$

$-2\sin A\sin B=\cos(A+B)-\cos(A-B)$

$a\sin a+b\cos a=\sqrt{a^2+b^2}\sin(a+c)$　$\tan C=\dfrac{b}{a}$

$\sin^2\alpha+\cos^2\alpha=1$　$1+\tan^2\alpha=\sec^2\alpha$

洛必达法则使用条件

1. 满足$\frac{0}{0}$型或$\frac{\infty}{\infty}$型

2. $f(x)$, $g(x)$ 在 x_0 去心邻域内可导，且 $g'(x)\neq 0$

3. $\lim\limits_{x\to x_0}\dfrac{f'(x)}{g'(x)}=a$（$a$为有限实数或者无穷大）则 $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to x_0}\dfrac{f'(x)}{g'(x)}=a$

$$\lim_{x\to\infty}\underbrace{\frac{\sin x+x}{x}}_{\text{极限存在}=1}=\lim_{x\to\infty}\underbrace{\frac{\cos x+1}{1}}_{\text{极限不存在}}$$

基本不等式

1. $a^2+b^2\geq 2ab$　$a,b\in R$，当且仅当$a=b$时等号成立。

2. $a+b\geq 2\sqrt{ab}$　$a,b\in[0,+\infty)$，当且仅当$a=b$时等号成立

3. $\frac{a^2+b^2}{2}\geq\left(\frac{a+b}{2}\right)^2\geq ab\geq\frac{2}{\frac{1}{a^2}+\frac{1}{b^2}}$　$a,b\in(0,+\infty)$，当且仅当$a=b$时等号成立

$\sqrt{\frac{a^2+b^2}{2}}\geq\frac{a+b}{2}\geq\sqrt{ab}\geq\frac{2}{\frac{1}{a}+\frac{1}{b}}$

算术-几何均值不等式

$$\frac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1\cdot a_2\cdots a_n}$$

分解：

$$x^3-1=(x-1)(x^2+x+1)$$

$$x^3+1=(x+1)(x^2-x+1)$$

求和：

1. 等比数列求和：

$$S_n=\begin{cases}\frac{a_1(1-q^n)}{1-q} & (q\neq 1)\\ na_1 & (q=1)\end{cases}$$

2. 等差数列求和：

(1) $S_n=\frac{n(a_1+a_n)}{2}$

(2) $S_n=na_1+\frac{n(n-1)}{2}d$

3. $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$

椭圆的切线方程

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$　点$P(x_0,y_0)$在椭圆上

则过P的椭圆的切线方程为$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$

椭圆的面积: $S=\pi ab$

周长: $L=2\pi b+4(a-b)$

b指的是短半径

1. 梯度

设 $u=f(x,y,z)$ 可偏导，则 $\mathbf{grad}u=\left\{\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right\}$.

2. 旋度

设向量场 $\vec{A}=\{P(x,y,z),Q(x,y,z),R(x,y,z)\}$，则 $\mathbf{rot}\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$.

3. 散度

设向量场 $\vec{A}=\{P(x,y,z),Q(x,y,z),R(x,y,z)\}$，则 $\mathrm{div}\vec{A}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$.

第一章 极限与连续.

Part I 极限.

一、defs

一1、极限

case1. (ε-N) 若 $\forall \varepsilon>0, \exists N>0$ 当 $n>N$ 时

$|a_n-A|<\varepsilon$

$\lim\limits_{n\to\infty} a_n=A$

如 $a_n=\frac{n}{2n+1}$ $\lim\limits_{n\to\infty}\frac{n}{2n+1}=\frac{1}{2}$ 但 $\frac{n}{2n+1}\neq\frac{1}{2}$

notes:

① $x\to a$, 则 $x\neq a$ 如: $\lim\limits_{x\to 0}\frac{0}{x^3}=0$

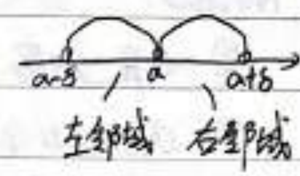
② $\lim\limits_{x\to a} f(x)$ 与 $f(a)$ 无关

如 $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}=\lim\limits_{x\to 1}(x+1)=2$

③ $x\to a \begin{cases} x\to a^- \\ x\to a^+ \end{cases}$

④ $0<|x-a|<\delta$

a的去心邻域



case2. (ε-δ) 若 $\forall \varepsilon>0, \exists \delta>0$ 当 $0<|x-a|<\delta$ 时

$|f(x)-A|<\varepsilon$

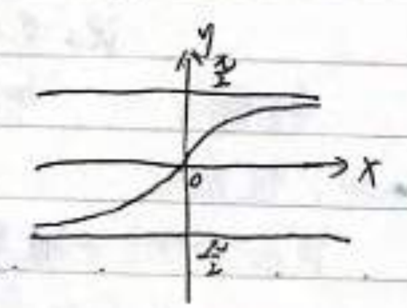
$\lim\limits_{x\to a} f(x)=A$

⑤ $\lim\limits_{x\to a^-} f(x) \triangleq f(a-0)$ — 左极限

$\lim\limits_{x\to a^+} f(x) \triangleq f(a+0)$ — 右极限

☆ $\lim\limits_{x\to a} f(x)\ \exists \Leftrightarrow f(a-0)、f(a+0)\ \exists$ 且相等.

· 如 $y=f(x)=\arctan x$

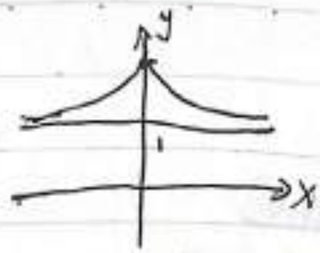


$\lim\limits_{x\to-\infty} f(x)=-\frac{\pi}{2}$

$\lim\limits_{x\to+\infty} f(x)=\frac{\pi}{2}$

又如：$y = 1 + e^{-x^2}$

$\lim\limits_{x\to\infty}(1+e^{-x^2}) = 1$



Case 3：$(\varepsilon - X)$ $\begin{cases} x\to+\infty \\ x\to-\infty \\ x\to\infty \end{cases}$

① If $\forall\varepsilon>0$. $\exists X>0$ 当 $x>X$ 时

$|f(x)-A|<\varepsilon$

$\lim\limits_{x\to+\infty} f(x) = A$

② 若 $\forall\varepsilon>0$，$\exists X>0$ 当 $\underline{x<-X}$ 时

$|f(x)-A|<\varepsilon$ $\quad |x|>X$

$\lim\limits_{x\to-\infty} f(x) = A$

2、无穷小

— 若 $\lim\limits_{x\to a}\alpha(x) = 0$

称 $\alpha(x)$ 当 $x\to a$ 时为无穷小

Notes：

① 0是无穷小，但无穷小不一定为0

② $\alpha(x)\neq 0$，$\alpha(x)$是否为无穷小与 x 趋向有关.

如，$\alpha = 3(x-1)^2$

$\lim\limits_{x\to1} 3(x-1)^2 = 0$ $\quad 3(x-1)^2$ 当 $x\to1$ 为无穷小.

设 $\alpha\to0$，$\beta\to0$

Case 1. $\lim\frac{\beta}{\alpha} = 0$ $\quad\beta$为α的高阶无穷小.

记 $\beta = o(\alpha)$

Case 2. $\lim\frac{\beta}{\alpha} = k$ $(\neq 0, \infty)$，β为α的同阶无穷小.

记 $\beta = O(\alpha)$

特例，$\lim\frac{\beta}{\alpha} = 1$ $\quad \alpha\sim\beta$

二、性质：

(一) 一般性质：

1.（唯一性）极限存在必唯一

证：设 $\lim\limits_{x\to a} f(x)=A$ ． $\lim\limits_{x\to a} f(x)=B$

(反)．设 $A>B$

取 $\varepsilon=\frac{A-B}{2}>0$

$\because \lim\limits_{x\to a} f(x)=A$ 　$\therefore \exists\, \delta_1>0$，当 $0<|x-a|<\delta_1$ 时

$|f(x)-A|<\frac{A-B}{2}$

$\Downarrow$

$\frac{A+B}{2}<f(x)<\frac{3A-B}{2}$ 　(*)

又 $\because \lim\limits_{x\to a} f(x)=B$ 　$\therefore \exists\, \delta_2>0$ 当 $0<|x-a|<\delta_2$ 时

$|f(x)-B|<\frac{A-B}{2}$

$\Downarrow$

$\frac{3B-A}{2}<f(x)<\frac{A+B}{2}$ 　(**)

取 $\delta=\min\{\delta_1,\delta_2\}$，当 $0<|x-a|<\delta$ 时．

(*)，(**) 成立 　矛盾 　$A>B$ 不对

同理 $A<B$ 也不对，$\therefore A=B$．

2（保号性）设 $\lim\limits_{x\to a} f(x)=A \begin{cases} >0 \\ <0 \end{cases}$

则 $\exists\, \delta>0$，当 $0<|x-a|<\delta$ 时

$f(x)\begin{cases} >0 \\ <0 \end{cases}$

证：设 $A>0$

取 $\varepsilon=\frac{A}{2}>0$

$\because \lim\limits_{x\to a} f(x)=A$ ，$\therefore \exists\, \delta>0$，当 $0<|x-a|<\delta$ 时．

$|f(x)-A|<\frac{A}{2} \Rightarrow f(x)>\frac{A}{2}>0$

设 $A<0$

取 $\varepsilon=-\frac{A}{2}>0$

$\because \lim\limits_{x\to a} f(x)=A$，$\therefore \exists\, \delta>0$，当 $0<|x-A|<\delta$ 时．

$|f(x)-A|<-\frac{A}{2} \Rightarrow f(x)<\frac{A}{2}<0$

例1. $f'(x)$ $f'(1)=0$ $\lim\limits_{x\to1}\frac{f'(x)}{(x-1)^3}=-2$, $x=1$?

解: $\because \lim\limits_{x\to1}\frac{f'(x)}{(x-1)^3}=-2<0$, $\therefore \exists\,\delta>0$ 当 $0<|x-1|<\delta$ 时

$\frac{f'(x)}{(x-1)^3}<0$

$$\begin{cases} f'(x)>0, & x\in(1-\delta,1) \\ f'(x)<0, & x\in(1,1+\delta) \end{cases}$$

$x=1$ 为极大点

(二) 存在性

准则I (数列型).

If ① $a_n\le b_n\le c_n$; ② $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}c_n=A$

$\Rightarrow \lim\limits_{n\to\infty}b_n=A$

准则 I′ (函数型)

If ① $f(x)\le g(x)\le h(x)$; ② $\lim f(x)=\lim h(x)=A$

$\Rightarrow \lim g(x)=A$

证 设 $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}h(x)=A$

$\forall\,\varepsilon>0$ $\because \lim\limits_{x\to a}f(x)=A$ $\therefore \exists\,\delta_1>0$ 当 $0<|x-a|<\delta_1$ 时

$|f(x)-A|<\varepsilon$

$\Updownarrow$

$A-\varepsilon<f(x)<A+\varepsilon$ (*)

又 $\because \lim\limits_{x\to a}h(x)=A$ $\therefore \exists\,\delta_2>0$, 当 $0<|x-a|<\delta_2$ 时.

$|h(x)-A|<\varepsilon$

$\Updownarrow$

$A-\varepsilon<f(x)<A+\varepsilon$ (*)

$A-\varepsilon<h(x)<A+\varepsilon$ (**)

取 $\delta=\min\{\delta_1,\delta_2\}$ 当 $0<|x-a|<\delta$ 时.

(*) (**) 成立.

$\therefore$ $A-\varepsilon<f(x)$ $h(x)<A+\varepsilon$

$\Rightarrow A-\varepsilon<g(x)<A+\varepsilon \Leftrightarrow |g(x)-A|<\varepsilon$

$\therefore \forall \varepsilon>0$. $\exists \delta>0$. 当 $0<|x-a|<\delta$ 时.

$|g(x)-A|<\varepsilon$

$\therefore \lim_{x\to a} g(x)=A$

型一. n项和求极限

例1. $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)$ (分母不齐) 用夹逼

例2. $\lim_{n\to\infty}\left(\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n}\right)$ (不齐)

分子、分母不齐 用夹逼

Note: 若分子齐. 分母非齐. 且分母多一次

定积分定义

动脑筋

记 $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{i}{n}\right)=\int_0^1 f(x)dx$

Note: 若分子齐、分母齐、且分母多一次 $\Downarrow$ 用定积分定义

例3. $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$

例4. $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right)$

准则Ⅱ. 单调有界数列必有极限

Notes:

① 有界 — If $M>0$. 使 $|a_n|\le M$

② $a_n\ge M$ — 有下界; $a_n\le M_2$ — 有上界

如: $|a_n|\le 3 \Rightarrow a_n\ge -3$. $a_n\le 3$

又如: $a_n\ge -2$. $a_n\le 4 \Rightarrow |a_n|\le 4$

$\{a_n\}$ 有界 $\Leftrightarrow$ 有上下界

③ Case1. $\{a_n\}\uparrow \begin{cases} \text{无上界} \Rightarrow \lim_{n\to\infty} a_n=+\infty \\ a_n\le M \Rightarrow \lim_{n\to\infty} a_n \exists \end{cases}$

Case2. $\{a_n\}\downarrow \begin{cases} \text{无下界} \Rightarrow \lim_{n\to\infty} a_n=-\infty \\ a_n\ge M \Rightarrow \lim_{n\to\infty} a_n \exists \end{cases}$

型二 极限存在证明

1. $a_1=\sqrt{2}$，$a_2=\sqrt{2+\sqrt{2}}$，$a_3=\sqrt{2+\sqrt{2+\sqrt{2}}}$，…

证：$\lim\limits_{n\to\infty} a_n \exists$，求之

证 $a_{n+1}=\sqrt{2+a_n}$

显然 $\{a_n\}\uparrow$

现证 $a_n\le 2$ 用数学归纳法.

2. $a_1=2$ $a_{n+1}=\frac{1}{2}(a_n+\frac{1}{a_n})$

证：$\lim\limits_{n\to\infty} a_n \exists$

证：$a_{n+1}>1$

$a_{n+1}-a_n=\frac{1}{2}(a_n+\frac{1}{a_n})-a_n=\frac{1}{2}(\frac{1}{a_n}-a_n)$

$=\frac{1-a_n^2}{2a_n}\le 0 \Rightarrow \{a_n\}\downarrow$

$\therefore \lim\limits_{n\to\infty} a_n \exists$

(三) 无穷小性质

1、一般性质

① $\alpha\to 0,\ \beta\to 0 \Rightarrow \begin{cases}\alpha\pm\beta\to 0\\ k\alpha\to 0\\ \alpha\beta\to 0\end{cases}$

② $|\alpha|\le M,\ \beta\to 0 \Rightarrow \alpha\beta\to 0$

如：$\lim\limits_{x\to 0} x^2\sin\frac{1}{x}=0$

③ $\lim f(x)=A \Leftrightarrow f(x)=A+\alpha\quad \alpha\to 0$

2、等价性质

① $\begin{cases}\alpha\sim\alpha\\ \alpha\sim\beta \Rightarrow \beta\sim\alpha\\ \alpha\sim\beta,\ \beta\sim\gamma \Rightarrow \alpha\sim\gamma\end{cases}$

② $\alpha\sim\alpha_1,\ \beta\sim\beta_1$，且 $\lim\frac{\beta_1}{\alpha_1}=A$

则 $\lim\frac{\beta}{\alpha}=A$

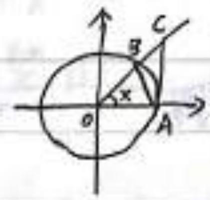
3. $x \to 0$ 时

① $x \sim \sin x \sim \tan x \sim \arcsin x \sim \arctan x$
$\sim e^x - 1 \sim \ln(x+1)$;

② $1 - \cos x \sim \frac{1}{2}x^2$

③ $(1+x)^a - 1 \sim ax$

三、两个重要极限



$0 < x < \frac{\pi}{2}$ 时

$S_{\triangle AOB} = \frac{1}{2}\sin x$

$S_{扇AOB} = \frac{1}{2}x$

$S_{\text{Rt}\triangle AOC} = \frac{1}{2}\tan x$

记: $\sin x < x < \tan x$

(I) $\lim_{\Delta \to 0} \frac{\sin \Delta}{\Delta} = 1$

(II) $\lim_{\Delta \to 0} (1+\Delta)^{\frac{1}{\Delta}} = e$

★ 型 = 不定型　$\frac{0}{0}$, 1^∞
$\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, ∞^0, 0^0

$\frac{0}{0}$ 型:

① 习惯

$$u(x)^{v(x)} \Rightarrow e^{v(x)\ln u(x)}$$

$$\ln(\quad) \Rightarrow \ln(1+\Delta) \sim \Delta \quad (\Delta \to 0)$$

$$(\quad) - 1 \Rightarrow \begin{cases} e^\Delta - 1 \sim \Delta \\ (1+\Delta)^a - 1 \sim a\Delta \end{cases} \quad (\Delta \to 0)$$

$x - \ln(1+x)$ — 二阶无穷小

x, $\sin x$, $\tan x$, $\arcsin x$, $\arctan x$ 任两个之差为3阶无穷小.

②误区：$\begin{cases} \lim\limits_{x\to 0} \dfrac{\sin 3x + (e^x-1)}{x} \\ \lim\limits_{x\to 0} \dfrac{e^{x^2}-\cos x}{x\ln(1+x)} \end{cases}$ 精确度一样

$\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$ （1 1） 精确度不一样 x^3 高

1、$\lim\limits_{x\to 0} \dfrac{\tan x-\sin x}{x^3} = \dfrac{\tan x}{x} \times \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

2、$\lim\limits_{x\to 0} \dfrac{(1+x^2)^{\sin x}-1}{x^2\ln(1+2x)}$

3、$\cancel{\lim\limits_{x\to 0} \dfrac{(1+\frac{\cos x}{2})^x-1}{x^3}}$ $\lim\limits_{x\to 0} \dfrac{(\frac{1+\cos x}{2})^x-1}{x^3}$

1^∞ 型：$(\ 1\ )^{(\infty)}$

$\begin{cases} \text{凑 } (1+\Delta)^{\frac{1}{\Delta}} \\ \text{恒等变形} \end{cases}$

1、$\lim\limits_{x\to 0} (1-x\sin x)^{\frac{1}{x-\ln(1+x)}}$

2、$\lim\limits_{x\to\infty} (\cos\frac{1}{x})^{x^2} = \lim\limits_{x\to\infty} \left\{[1+(\cos\frac{1}{x}-1)]^{\frac{1}{\cos\frac{1}{x}-1}}\right\}^{x^2(\cos\frac{1}{x}-1)}$

$= e^{\lim\limits_{x\to\infty} \frac{\cos\frac{1}{x}-1}{\frac{1}{x^2}}} \xlongequal{\frac{1}{x}=t} e^{\lim\limits_{t\to 0} \frac{\cos t-1}{t^2}} = e^{-\frac{1}{2}}$

3、$\lim\limits_{x\to 0} (\frac{1+x}{1+\sin x})^{\frac{1}{x^3}}$

$= \lim\limits_{x\to 0} \left[(1+\frac{x-\sin x}{1+\sin x})^{\frac{1+\sin x}{x-\sin x}}\right]^{\frac{x-\sin x}{1+\sin x}\cdot\frac{1}{x^3}} = e^{\lim\limits_{x\to 0} \frac{1}{1+\sin x} \times \frac{x-\sin x}{x^3}} = e^{\lim\limits_{x\to 0} \frac{1-\cos x}{3x^2}}$

$\infty-\infty$ 型

1、$\lim\limits_{x\to 0} (\frac{1}{x^2}-\frac{1}{\tan^2 x}) = \lim\limits_{x\to 0} \dfrac{\tan^2 x-x^2}{x^4}$

2、$\lim\limits_{x\to+\infty} (\sqrt{x^2-4x+8}-x)$

$= \lim\limits_{x\to+\infty} \dfrac{-4x+8}{\sqrt{x^2-4x+8}+x}$

$\frac{\infty}{\infty}$ 型：

① $\lim_{x\to\infty} \frac{b_m x^m + \cdots + b_0}{a_n x^n + \cdots + a_0} \begin{cases} =0 & , m<n \\ =\frac{b_m}{a_m} & , m=n \\ =\infty & , m>n \end{cases}$

例1. $\lim_{n\to\infty} \frac{n^{2019}}{(n+1)^a - n^a} = k \ (\neq 0, \infty)$ a, k?

解：$(n+1)^a = C_a^0 n^a + C_a^1 n^{a-1} + \cdots + C_a^a$

$= n^a + \cancel{a^{n-1}} a n^{a-1} + *$

$(n+1)^a - n^a = a n^{a-1} + *$

$\therefore \begin{cases} a-1 = 2019 \\ k = \frac{1}{a} \end{cases} \Rightarrow a = 2020, \ k = \frac{1}{2020}$

例2. $\lim_{x\to\infty} \frac{\ln(x^3+3)}{\ln(x^4+3x+1)} = \lim_{x\to+\infty} \frac{\ln x^3(1+\frac{3}{x^3})}{\ln x^4(1+\frac{3}{x^3}+\frac{1}{x^4})}$

② 罗比法则

1. $\lim_{x\to\infty} \frac{x^3}{e^x}$ 　　2. $\lim_{x\to+\infty} \frac{\ln x}{\sqrt{x}}$

速度 指数 > 幂 > 对数

$0\times\infty$ 型

$0\times\infty \Rightarrow \begin{cases} \frac{0}{\frac{1}{\infty}} & 即 \frac{0}{0} \\ \frac{\infty}{\frac{1}{0}} & 即 \frac{\infty}{\infty} \end{cases}$

1. $\lim_{x\to\infty} (2x+1)^2 \sin\frac{1}{x^2} = \lim_{x\to\infty} \frac{(2x+1)^2}{x^2} \frac{\sin\frac{1}{x^2}}{\frac{1}{x^2}}$

2. $\lim_{x\to\infty} [x - x^2 \ln(1+\frac{1}{x})]$ $(\infty-\infty)$ 转化

$= \lim_{x\to\infty} x^2 [\frac{1}{x} - \ln(1+\frac{1}{x})]$ $(\infty\times 0)$

$\left.\begin{matrix} \infty^0 \\ 0^0 \end{matrix}\right\} \Rightarrow e^{\ln}$

1. $\lim_{x\to 0^+} x^x = 1$

型五 左右极限

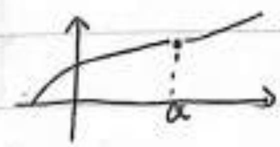
☆ $a^{\frac{?}{x-b}}$ 或 $a^{\frac{?}{b-x}}$ 当 $x\to b$ 一定分左右

如 $f(x)=e^{\frac{x}{x-2}}$; $\lim\limits_{x\to 2} f(x)$

例1. $f(x)=\dfrac{1-2^{\frac{1}{x-1}}}{1+2^{\frac{1}{x-1}}}$. $\lim\limits_{x\to 1} f(x)$

Part II 连续与间断

$\lim\limits_{x\to a} f(x)$ 与 $f(a)$ 无关



一、defs.

1.连续

① 若连续 — if $\lim\limits_{x\to a} f(x)=f(a)$

或 $f(a-0)=f(a+0)=f(a)$

得 $f(x)$ 在 $x=a$ 连续.

② $f(x)$ 在 $[a,b]$ 上连续

If $\begin{cases} f(x)\text{在}(a,b)\text{内连续} \\ f(a)=f(a+0),\ f(b)=f(b-0) \end{cases}$

得 $f(x)$ 在 $[a,b]$ 上连续　记 $f(x)\in C[a,b]$.

☆ 2.间断 — If $\lim\limits_{x\to a} f(x)\neq f(a)$

分类

① 第一类：$f(a-0)$, $f(a+0)$ ∃

$\begin{cases} f(a-0)=f(a+0)\ (\neq f(a)) & \text{— 可去间断点.} \\ f(a-0)\neq f(a+0) & \text{— a为跳跃间断点.} \end{cases}$

② 第二类：

$f(a-0)$, $f(a+0)$ 至少有一个无

型六 间断点的分类

1. $f(x)=\frac{x^2-3x+2}{x^2-1}e^{\frac{1}{x}}$

解. $x=0$, $x=\pm1$ 为间断点

$f(0-0)=0$, $f(0+0)=-\infty \Rightarrow x=0$ 为第二类间断点

$\lim_{x\to-1} f(x)=\infty \Rightarrow x=-1$ 为第二类间断点

$\lim_{x\to1} f(x)=\lim_{x\to1}\frac{x-2}{x+1}e^{\frac{1}{x}}=-\frac{e}{2} \Rightarrow x=1$ 为可去间断点

2. $f(x)=\frac{x e^{\frac{1}{x-1}}}{1-e^{\frac{x}{x-1}}}$

解: $x=0$, $x=1$ 为间断点

∵ $x\to0$ 时, $1-e^{\frac{x}{x-1}}\sim\frac{x}{1-x}$

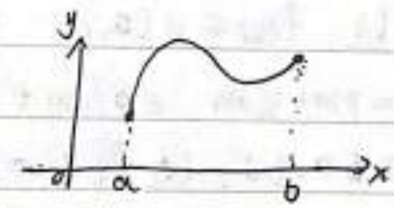
∴ $\lim_{x\to0} f(x)=\lim_{x\to0} e^{\frac{1}{x-1}}\times\frac{x}{1-e^{\frac{x}{x-1}}}=e^{-1}\lim_{x\to0}\frac{x}{\frac{x}{x-1}}=\frac{1}{e}$

∴ $x=0$ 为可去间断点

$f(1-0)=0 \neq f(1+0)=\underbrace{-\lim_{x\to1^+} e^{\frac{1-x}{x-1}}}=-\frac{1}{e}$

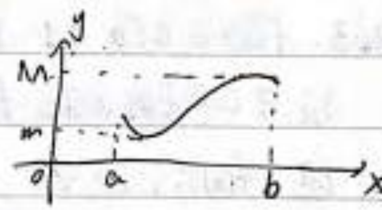
$x=1$ 为跳跃间断点

二、 $f(x)\in C[a,b]$: 闭区间连续



性质1 (最值)

$f(x)\in C[a,b]\Rightarrow \exists\, m, M$

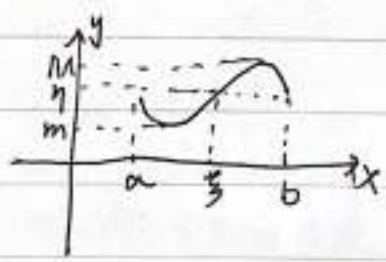


2. (有界)

$f(x)\in C[a,b]\Rightarrow \exists K>0$, $|f(x)|\le K$

3. (零点定理) $f(x)\in C[a,b]$ 且 $f(a)f(b)<0$

则 $\exists\, c\in(a,b)$, $f(c)=0$



介值 — the value between m and M

$\forall \eta\in[m,M]$

$\exists\, \xi\in\underbrace{[a,b]}_{闭}$, $f(\xi)=\eta$

4(介值) $f(x)\in C[a,b]$. $\forall \eta\in[m,M]$.

$\exists \xi\in[a,b]$，使 $f(\xi)=\eta$

(位于 m 和 M 之间的值皆可被 $f(x)$ 取到)

Notes:

① $f(x)\in C[a,b]$. $\exists c\in(a,b)$ ⇒ 零点

例1. 证：$x^5-4x+1=0$ 至少一个正根

证：令 $f(x)=x^5-4x+1\in C[0,1]$

② $f(x)\in C[a,b]$ $\begin{cases}\xi\in[a,b]\\ 函数值之和\end{cases}$ ⇒ 介

例2. $f(x)\in C[a,b]$，$p>0$. $q>0$. $p+q=1$.

证：$\exists\xi\in[a,b]$，$f(\xi)=pf(a)+qf(b)$

证：$f(x)\in C[a,b]$ ⇒ $\exists m, M$

$m=pm+qm\le pf(a)+qf(b)\le pM+qM=M$

$\therefore \exists\xi\in[a,b]$，使 $f(\xi)=pf(a)+qf(b)$

例3. $f(x)\in C[0,2]$ $f(0)+2f(1)+3f(2)=6$

证 $\exists c\in[0,2]$，$f(c)=1$

证：$f(x)\in C[0,2]$ ⇒ $\exists m, M$

$6m\le f(0)+2f(1)+3f(2)\le 6M$

$\therefore m\le 1\le M$

$\exists c\in[0,2]$，$f(c)=1$

第二章 导数与微分

一、defs.

1.导数

$y=f(x)\ (x\in D)$. $x_0\in D$

$x_0+\Delta x\in D$

$\Delta y = f(x_0+\Delta x)-f(x_0)$

If $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ ∃. 称 $f(x)$ 在 x_0 可导

$\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \triangleq f'(x_0)$. $\frac{dy}{dx}\Big|_{x=x_0}$

Notes:

①.等价定义: $x_0\to x \Rightarrow f(x_0)\to f(x)$.

$\Delta x = x-x_0$. $\Delta y = f(x)-f(x_0)$

$\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$. $\frac{dy}{dx}\Big|_{x=x_0}$

② $\Delta x\to 0\begin{cases}\Delta x\to 0^-\\ \Delta x\to 0^+\end{cases}$ 或 $x\to a\begin{cases}x\to a^-\\ x\to a^+\end{cases}$

$\lim\limits_{\Delta x\to 0^-}\frac{\Delta y}{\Delta x}\left(=\lim\limits_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}\right)\triangleq f_-'(x_0)$

$\lim\limits_{\Delta x\to 0^+}\frac{\Delta y}{\Delta x}\left(=\lim\limits_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}\right)\triangleq f_+'(x_0)$

$f'(x_0)$ ∃ $\Leftrightarrow$ $f_-'(x_0)$、$f_+'(x_0)$ ∃且相等.

③ $f(x)$在 x_0 可导 $\underset{\not\Leftarrow}{\Rightarrow}$ $f(x)$ 在 x_0 连续

"$\Rightarrow$" $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ ∃

$\Rightarrow \lim\limits_{x\to x_0}[f(x)-f(x_0)]=0$

$\Rightarrow \lim\limits_{x\to x_0}f(x)=f(x_0)$

"$\not\Leftarrow$"

例1. $f(x)=\begin{cases}e^x-1, & x<0\\ \ln(1+2x), & x\geq 0\end{cases}$ $f'(0)$?

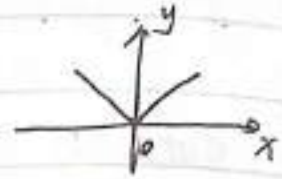
解: 1° $f(0-0)=0=f(0)=f(0+0)$ $\Rightarrow$ $f(x)$在$x=0$连续.

2° $f_-'(0)=\lim\limits_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^-}\frac{e^x-1}{x}=1$

$f_+'(0)=\lim\limits_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^+}\frac{\ln(1+2x)}{x}=2$

$\because f_-'(0)\neq f_+'(0)$ $\therefore f'(0)$不存在

例2. $f(x)=|x|$, $f'(0)$?



解:

$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{|x|}{x}=-1$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{|x|}{x}=1$$

④ $f(x)$ 连续, 若 $\lim_{x\to a}\frac{f(x)-b}{x-a}=A \Rightarrow f(a)=b \quad f'(a)=A$

$$\lim_{x\to a}\frac{f(x)-b}{x-a}=A \Rightarrow \lim_{x\to a}[f(x)-b]=0 \Rightarrow \lim_{x\to a}f(x)=b \Rightarrow f(a)=b$$

$$A=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

2. 可微 — $y=f(x)$ $(x\in D)$. $x_0\in D$

$x_0\to x_0+\Delta x$, $\quad f(x_0)\longrightarrow f(x_0+\Delta x)$

$\Delta y=f(x_0+\Delta x)-f(x_0)\overset{if}{=\!=}A\Delta x+o(\Delta x)$ 称 $f(x)$ 在 x_0 可微

$A dx = A\Delta x \triangleq dy|_{x=x_0}$ —— 微分

Notes: ① 可导 $\Leftrightarrow$ 可微

"$\Rightarrow$" $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(x_0) \Rightarrow \frac{\Delta y}{\Delta x}=f'(x_0)+\alpha \quad \alpha\to 0\ (\Delta x\to 0)$

$\Rightarrow \Delta y=f'(x_0)\Delta x+\alpha\Delta x$

$\because \lim_{\Delta x\to 0}\frac{\alpha\Delta x}{\Delta x}=0 \quad \therefore \alpha\Delta x=o(\Delta x)$

$\Delta y=f'(x_0)\Delta x+o(\Delta x)$

"$\Leftarrow$" $\Delta y=A\Delta x+o(\Delta x) \Rightarrow \frac{\Delta y}{\Delta x}=A+\frac{o(\Delta x)}{\Delta x}$

$\because \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=A \quad \therefore f(x)$ 在 x_0 可导 且 $f'(x_0)=A$.

② If $\Delta y=A\Delta x+o(\Delta x)$ 则 $A=f'(x_0)$

③ 设 $y=f(x)$ 可导 则 $dy=df(x)=f'(x)\,dx$

如: $d(x^3)=3x^2dx$

二. 求导工具:

(一) 求导公式

1. $(C)'=0$
2. $(x^a)'=ax^{a-1}$
3. $(a^x)'=a^x\ln a \qquad (e^x)'=e^x$

4. $(\log_a x)' = \frac{1}{x\ln a}$ $(\ln x)' = \frac{1}{x}$

5. ① $(\sin x)' = \cos x$

② $(\cos x)' = -\sin x$

③ $(\tan x)' = \sec^2 x$

④ $(\cot x)' = -\csc^2 x$

⑤ $(\sec x)' = \sec x \tan x$

⑥ $(\csc x)' = -\csc x \cot x$

6. ① $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

② $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

③ $(\arctan x)' = \frac{1}{1+x^2}$

④ $(\text{arccot}\, x)' = -\frac{1}{1+x^2}$

(二) 四则：

1. $(u \pm v)' = u' \pm v'$

2. ① $(ku)' = ku'$

② $(uv)' = u'v + uv'$

③ $(uvw)' = u'vw + uv'w + uvw'$

例. $f(x) = x(x+1)\cdots(x+99)$ 求 $f'(0)$

法一：$f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} (x+1)\cdots(x+99) = 99!$

法二：$f'(x) = (x+1)\cdots(x+99) + x(x+2)\cdots(x+99) + \cdots + x(x+1)\cdots(x+98)$

$f'(0) = 99!$

3. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

(三) 复合：

Th. $y=f(u)$ 可导 $u=\varphi(x)$ 可导 且 $\varphi'(x) \neq 0$

则 $y = f[\varphi(x)]$ 可导，且

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot \varphi'(x) = f'[\varphi(x)] \cdot \varphi'(x)$

证：$\varphi'(x) = \lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \neq 0 \Rightarrow \Delta u = O(\Delta x)$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\cdot\varphi'(x)$$
$$=f'(u)\cdot\varphi'(x)=f'[\varphi(x)]\cdot\varphi'(x)$$

例1. $y=\ln(\arctan x^2)$，求 $\frac{dy}{dx}$.

解：$y'=\frac{1}{\arctan x^2}\times\frac{1}{1+x^4}\times 2x$

例2. $y=e^{\sin^2\frac{1}{x}}$ 求 y'

解：$y'=e^{\sin^2\frac{1}{x}}\times 2\sin\frac{1}{x}\times\cos\frac{1}{x}\times\left(-\frac{1}{x^2}\right)$

(四) 反函数导数：

中学：$y=f(x)\Rightarrow x=\varphi(y)\Rightarrow y=\varphi(x)$

大学：$y=f(x)\Rightarrow x=\varphi(y)$

例3. 求 $y=\ln(x+\sqrt{1+x^2})$ 反函数

解：$y=\ln(x+\sqrt{1+x^2})\Rightarrow x+\sqrt{1+x^2}=e^y$

$\because -x+\sqrt{1+x^2}=e^{-y}$

$\therefore$ 反 $x=\frac{e^y-e^{-y}}{2}$

Th. $y=f(x)$ 可导 且 $f'(x)\neq 0$. 反函数为 $x=\varphi(y)$

则 $\varphi'(y)=\frac{1}{f'(x)}$

证：$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\neq 0\Rightarrow \Delta y=O(\Delta x)$

$$\varphi'(y)=\lim_{\Delta y\to 0}\frac{\Delta x}{\Delta y}=\lim_{\Delta y\to 0}\frac{1}{\frac{\Delta y}{\Delta x}}=\frac{1}{\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}}=\frac{1}{f'(x)}$$

型：

Case1: 显函数求导

1. $y=e^{2x}\sin x+\ln^2\tan(e^x+x^2)$ 求 y'

解：$y'=2e^{2x}\sin x+e^{2x}\cos x+2\ln\tan(e^x+x^2)\times\frac{1}{\tan(e^x+x^2)}\times \sec^2(e^x+x^2)\ \cancel{\frac{1}{1+(e^x+x^2)^2}}$
$\times(e^x+2x)$

Case2: 复合函数求导

Ex：$y=\arctan^2\frac{1-x}{1+x}$

$y'=2\arctan\frac{1-x}{1+x}\times\frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}\times\frac{-(1+x)-(1-x)}{(1+x)^2}$

Case 3. 隐函数求导

$y=f(x)$ 一显

$F(x,y)=0$ 一隐函数

$\overset{\text{出理谢}}{y}=f(x)$ 一隐函数显式化

$F(x, \underset{\varphi(x)}{y})=0$

例1. $e^{xy}=x^2+y^2+1$ 求 $\frac{dy}{dx}$

解: $e^{xy}=x^2+y^2+1$ 两边对 x 求导

$e^{xy}\cdot(y+x\frac{dy}{dx})=2x+2y\times\frac{dy}{dx}$

2. $2^{xy}+x^2=y$. 求 $dy|_{x=0}$

解, 1°. $x=0 \Rightarrow y=1$

2°. $2^{xy}+x^2=y$ 两边对 x 求导

$\Rightarrow 2^{xy}\cdot\ln2\cdot(y+x\frac{dy}{dx})+2x=\frac{dy}{dx}$

3°. $x=0$, $y=1$ 代入.

$2^0\ln2\times1+2\times0=y'(0) \Rightarrow y'(0)=\ln2$

$dy|_{x=0}=\ln2\,dx$

3. $\arctan\frac{y}{x}=\ln\sqrt{x^2+y^2}$, 求 $\frac{dy}{dx}$

解: $\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{xy'-y}{x^2}=\frac{1}{\sqrt{x^2+y^2}}\times\frac{1}{2\sqrt{x^2+y^2}}\times(2x+2y\cdot y')$

$\Rightarrow xy'-y=x+yy'$

Case 4. 参数方程

① $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$ $\frac{dy}{dx}$. $\frac{d^2y}{dx^2}$

例1. $\begin{cases} x=\arctan t \\ y=\ln(1+t^2) \end{cases}$ $\frac{d^2y}{dx^2}$

解: $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2t}{1+t^2}\Big/\frac{1}{1+t^2}=2t$

$(\frac{d^2y}{dx^2}\overset{?}{=}2 \times)$

$\frac{d^2y}{dx^2}=\frac{d(\frac{dy}{dx})}{dx}=\frac{d(2t)/dt}{dx/dt}=\frac{2}{\frac{1}{1+t^2}}=2(1+t^2)$

例2. $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$ $\quad \dfrac{d^2y}{dx^2}$?

解: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\sin t}{1-\cos t}$

$\dfrac{d^2y}{dx^2} = \dfrac{d(\frac{dy}{dx})}{dx} = \dfrac{d(\frac{\sin t}{1-\cos t})/dt}{dx/dt} = \dfrac{(\frac{\sin t}{1-\cos t})'}{1-\cos t}$

② $\begin{cases} f(x,t) = 0 \\ g(y,t) = 0 \end{cases}$ $\quad \dfrac{dy}{dx}$

例3. $\begin{cases} \sin(x+t) = xt \\ e^{ty} = y+t+1 \end{cases}$ $\quad \dfrac{dy}{dx}$

解: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$\cos(x+t)\cdot(\frac{dx}{dt}+1) = x + t\frac{dx}{dt}$

$e^{ty}(y + t\frac{dy}{dt}) = \frac{dy}{dt} + 1$

Case 5. 分段函数求导

1. $f(x) = \begin{cases} e^{ax}, & x<0 \\ \ln(1+3x)+b, & x\geq 0 \end{cases}$ $\quad f'(0)\exists$, $\quad a, b$?

解: 1° $f(0-0) = 1$

$f(0) = f(0+0) = b$

$\because f(x)$在$x=0$连续 $\quad \therefore b=1$

$f(x) = \begin{cases} e^{ax}, & x<0 \\ \ln(1+3x)+1, & x\geq 0 \end{cases}$

2° $f'_-(0) = \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \dfrac{e^{ax}-1}{x} = a$

$f'_+(0) = \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \dfrac{\ln(1+3x)}{x} = 3$

$\because f'_-(0) = f'_+(0) \quad \therefore a=3$

Case 6. 高阶导数.

① 归纳法.

1. $y = e^x \sin x \quad y^{(n)}$.

解: $y' = e^x\sin x + e^x\cos x = \sqrt{2}e^x \sin(x+\frac{\pi}{4})$

$y^{(n)} = (\sqrt{2})^n e^x \sin(x+\frac{n\pi}{4})$

2. $y=\frac{1}{2x+3}$, $y^{(n)}$.

解: $y=(2x+3)^{-1}$ $\quad y'=(-1)(2x+3)^{-2}\times 2$

$y''=(-1)(-2)(2x+3)^{-3}\times 2^2$

$y^{(n)}=\dfrac{(-1)^n n!\cdot 2^n}{(2x+3)^{n+1}}$

3. $y=\cancel{x^2}\ \frac{1}{x^2-1}$. $y^{(n)}$ $\quad \left(\frac{1}{ax+b}\right)^{(n)}=\dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

解: $y=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$

$y^{(n)}=\frac{1}{2}\left[\dfrac{(-1)^n n!}{(x-1)^{n+1}}-\dfrac{(-1)^n n!}{(x+1)^{n+1}}\right]$

② 公式法.

记:

$$(\sin x)^{(n)}=\sin\left(x+\frac{n\pi}{2}\right)$$

$$(\cos x)^{(n)}=\cos\left(x+\frac{n\pi}{2}\right)$$

$$(uv)^{(n)}=C_n^0 u^{(n)}v+C_n^1 u^{(n-1)}v'+\cdots+C_n^n u v^{(n)}$$

1. $y=x^2e^{3x}$ $\quad y^{(5)}$

解: $y=C_5^0 x^2(e^{3x})^{(5)}+C_5^1\, 2x(e^{3x})^{(4)}+C_5^2\, 2\,(e^{3x})^{(3)}$

$=3^5x^2\cdot e^{3x}+10\times 3^4 x\, e^{3x}+20\times 3^3\times e^{3x}$

第三章. 中值定理与一元微分学应用

$\boxed{\text{Part I}\quad \text{中值定理}}$

预备知识:

1. $f'(a)\begin{cases} >0 \\ <0 \\ =0 \\ \text{无}\end{cases}$

If $f'(a)>0$

$f'(a)\triangleq\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}>0$

$\delta>0$, 当 $0<|x-a|<\delta$ 时. $\dfrac{f(x)-f(a)}{x-a}>0$

$\begin{cases} f(x)<f(a) & , x\in(a-\delta,a) \\ f(x)>f(a) & , x\in(a,a+\delta)\end{cases}$

即 $f'(a)>0$ $\Rightarrow$ 左小右大. (a不是极值点)

If $f'(a)<0$

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}<0$$

$\exists\,\delta>0$ 当 $0<|x-a|<\delta$ 时 $\frac{f(x)-f(a)}{x-a}<0$

$$\begin{cases} f(x)>f(a) & x\in(a-\delta,a) \\ f(x)<f(a) & x\in(a,a+\delta) \end{cases}$$

$f'(a)<0 \Rightarrow$ 左大右小，(a不是极值点)

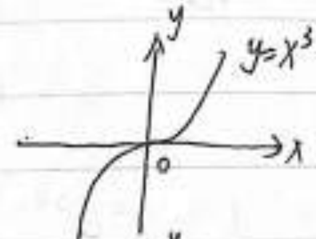
结论：

① $f(x)$在$x=a$取极值 $\underset{\not\Leftarrow}{\Rightarrow}$ $f'(a)=0$ 或 $f'(a)$无

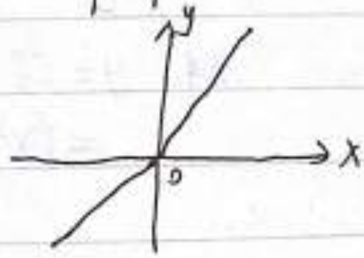
② $f(x)$可导且在$x=a$取极值 $\overset{\not\Leftarrow}{\Rightarrow}$ $f'(a)=0$

反1. $y=f(x)=x^3$.

$f'(x)=3x^2$, $f'(0)=0$



反2. $f(x)=\begin{cases} x, & x<0 \\ 2x, & x\geq 0 \end{cases}$

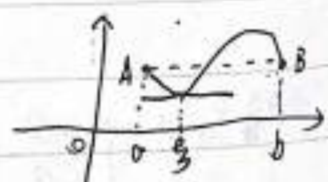


$f'_-(0)=1 \neq f'_+(0)=2$

$f'(0)$不存在.

Th1 (Rolle) If
罗尔

$$\begin{cases} f(x)\in C[a,b] \\ f(x)\text{在}(a,b)\text{内可导} \\ f(a)=f(b) \end{cases}$$



则$\exists\, \xi\in(a,b)$，使$f'(\xi)=0$

证：$f(x)\in C[a,b] \Rightarrow \exists\, m, M$.

① $m=M$. $f(x)\equiv C_0$. $\forall \xi\in(a,b)$, $f'(\xi)=0$;

② $m<M$

$\because f(a)=f(b)$

$\therefore m, M$至少一个在(a,b)内取

设 $\exists\, \xi \in (a,b)$， $f(\xi)=m$.

$\Rightarrow f'(\xi)=0$ 或 $f'(\xi)$ 不存在

而 $f(x)$ 在 (a,b) 内可导.

$\therefore f'(\xi)=0$

Th2. (lagrange) $\begin{cases} f(x) \in C[a,b] \\ f(x) \text{ 在 } (a,b) \text{ 内可导.} \end{cases}$

则 $\exists\, \xi \in (a,b)$，使 $f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$

分析： $L: y=f(x)$　　 $L_{AB}: \cancel{y=}\ y-f(a)=\dfrac{f(b)-f(a)}{b-a}(x-a)$

即: $L_{AB}: y=f(a)+\dfrac{f(b)-f(a)}{b-a}(x-a)$

辅助函数

证：令 $\varphi(x) = \text{曲} - \text{直} = f(x)-f(a)-\dfrac{f(b)-f(a)}{b-a}(x-a)$

$\varphi(x) \in C[a,b]$. $\varphi(x)$ 在 (a,b) 内可导.

$\varphi(a)=\varphi(b)=0$

$\exists\, \xi \in (a,b)$，使 $\varphi'(\xi)=0$

而 $\varphi'(x) = f'(x)-\dfrac{f(b)-f(a)}{b-a}$

$\therefore f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$

Notes:

① If $f(a)=f(b)$，则 $L \Rightarrow R$

② 等价形式

$$f'(\xi)=\frac{f(b)-f(a)}{b-a} \Leftrightarrow f(b)-f(a)=f'(\xi)(b-a)$$

$$\Leftrightarrow f(b)-f(a)=f'[a+\theta(b-a)](b-a) \quad (0<\theta<1)$$

Th3. (Cauchy) $\begin{cases} f、g \in C[a,b] \\ f、g \text{ 在 } (a,b) \text{ 内可导} \\ g'(x) \neq 0 \quad (a<x<b) \end{cases}$

则 $\exists\, \xi \in (a,b)$，使得 $\dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(\xi)}{g'(\xi)}$

Notes:

① $g'(x)\neq 0\ (a<x<b) \Rightarrow \begin{cases} g'(\xi)\neq 0 \\ g(b)-g(a)\neq 0 \end{cases}$

② If $g(x)=x \Rightarrow C \to L$

③ 辅助函数:

L: $\varphi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$

C: $\varphi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$

证: 令 $\varphi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$

$\varphi(x) \in C[a,b]$. (a,b) 内可导

$\varphi(a) = \varphi(b) = 0$

$\exists \xi \in (a,b)$. 使 $\varphi'(\xi)=0$

而 $\varphi'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x)$

$\therefore f'(\xi) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(\xi) = 0$

型一 $f^{(n)}(\xi)=0$

1. $f(x) \in C[a,b]$. (a,b) 内可导.

$f(a)f(b)>0$. $f(a)f(\frac{a+b}{2})<0$

证: $\exists \xi \in (a,b)$. $f'(\xi)=0$

证: 1° $f(a)f(\frac{a+b}{2})<0 \Rightarrow \exists x_1 \in (a, \frac{a+b}{2})$. $f(x_1)=0$

2° $f(b)f(\frac{a+b}{2})<0 \Rightarrow \exists x_2 \in (\frac{a+b}{2}, b)$. $f(x_2)=0$

3° $f(x_1)=f(x_2)=0$. $\exists \xi \in (x_1, x_2) \subset (a,b)$. $f'(\xi)=0$

2. $f(x) \in C[0,3]$, 在 $(0,3)$ 内二阶可导.

$3f(0) = f(1) + 2f(2) = 6$. $f(3)=2$

原函数找三等

证: $\exists \xi \in (0,3)$, $f''(\xi)=0$

1° $f(0)=2$, $f(3)=2$

2° $f(x) \in C[1,2] \Rightarrow f(x)$ 在 $[1,2]$ 上有 m, M.

$3m \le f(1) + 2f(2) \le 3M$

$\therefore m \le 2 \le M$

$\exists c \in [1,2]$. 使 $f(c)=2$

3° $f(0) = f(c) = f(3) = 2$.

$\exists \xi_1 \in (0, c)$, $\xi_2 \in (c, 3)$. $f'(\xi_1)=0$, $f'(\xi_2)=0$.

4° $\because f'(\xi_1) = f'(\xi_2) = 0$

$\therefore \exists \xi \in (\xi_1, \xi_2) \subset (a, b)$, 使 $f''(\xi)=0$

3. $f(x) \in C[0,1]$，$(0,1)$ 内二阶可导.

求一阶导两相等

$f(0)=0$，$f'(0)=2$，$f(1)=2$ 证 $\exists \xi \in (0,1)$，$f''(\xi)=0$.

证：1° $\exists c \in (0,1)$，使 $f'(c) = \frac{f(1)-f(0)}{1-0} = 2$

2° $\because f'(0) = f'(c) = 2$

$\therefore \exists \xi \in (0, c) \subset (0,1)$，$f''(\xi)=0$

型二，仅有 ξ，无其他字母

①还原法.

$$\frac{f'(x)}{f(x)} = [\ln f(x)]'$$

1. $f(x) \in C[0,1]$，$(0,1)$ 内可导，$f(1)=0$

证 $\exists \xi \in (0,1)$，使 $\xi f'(\xi) + 2f(\xi) = 0$

分析： $x f'(x) + 2f(x) = 0 \Rightarrow \frac{f'(x)}{f(x)} + \frac{2}{x} = 0$

$\Rightarrow [\ln f(x)]' + (\ln x^2)' = 0$

$\Rightarrow [\ln \underline{x^2 f(x)}]' = 0$

证：令 $\varphi(x) = x^2 f(x)$

$\varphi(0) = \varphi(1) = 0$

$\exists \xi \in (0,1)$，使 $\varphi'(\xi) = 0$

而 $\varphi'(x) = 2x f(x) + x^2 f'(x)$

$\therefore 2\xi f(\xi) + \xi^2 f'(\xi) = 0$

$\because \xi \neq 0$

$\therefore 2f(\xi) + \xi f'(\xi) = 0$

2. $f(x) \in C[a,b]$，(a,b)内可导，$f(a)=f(b)=0$

证：$\exists\ \xi \in (a,b)$，$f'(\xi)-2f(\xi)=0$

分析：$f'(x)-2f(x)=0 \Rightarrow \frac{f'(x)}{f(x)}-2=0$

$\Rightarrow [\ln f(x)]' + (\ln e^{-2x})' = 0$

$\Rightarrow [\ln e^{-2x} f(x)]' = 0$

证：令 $\varphi(x) = e^{-2x} f(x)$

$\varphi(a) = \varphi(b) = 0$

$\exists\ \xi \in (a,b)$，$\varphi'(\xi)=0$

而 $\varphi'(x) = -2e^{-2x}f(x) + e^{-2x}f'(x)$

$= e^{-2x}[f'(x)-2f(x)]$

$\therefore e^{-2\xi}[f'(\xi)-2f(\xi)]=0$

$\because e^{-2\xi} \neq 0 \quad \therefore f'(\xi)-2f(\xi)=0$

3. $f(x) \in C[0,1]$，$f(x)$在$(0,1)$内二阶可导

$f(0)=f(1)$，证$\exists\ \xi \in (0,1)$使 $f''(\xi) = \frac{2f'(\xi)}{1-\xi}$

分析：$\frac{f''(x)}{f'(x)} + \frac{2}{x-1} = 0$

$\Rightarrow [\ln f'(x)]' + [\ln(x-1)^2]' = 0$

证：令 $\varphi(x) = (x-1)^2 f'(x)$

$\varphi(1) = 0$

$f(0)=f(1) \Rightarrow \exists\ c \in (0,1)$，$f'(c)=0$

$\varphi(c)=0$

$\because \varphi(c) = \varphi(1) = 0$

$\therefore \exists\ \xi \in (c,1) \subset (0,1)$，使 $\varphi'(\xi)=0$

而 $\varphi'(x) = 2(x-1)f'(x) + (x-1)^2 f''(x)$

$\therefore 2(\xi-1)f'(\xi) + (\xi-1)^2 f''(\xi) = 0$

$\because \xi \neq 1 \quad \therefore \frac{2f'(\xi)}{\xi-1} + f''(\xi) = 0$

型三．有ξ，有a,b.

Case1．a,b与ξ可分

方法：ξ与a,b分离 $\Rightarrow$ a,b侧 $\begin{cases} \frac{f(b)-f(a)}{b-a} & —L格 \\ \frac{f(b)-f(a)}{g(b)-g(a)} & —C. \end{cases}$

1. $f(x)\in C[a,b]$，(a,b)内可导 $(a>0)$

证 $\exists\xi\in(a,b)$，$f(b)-f(a)=\xi f'(\xi)\cdot\ln\frac{b}{a}$

分析：$\frac{f(b)-f(a)}{\ln b-\ln a}=\xi f'(\xi)$

证：令$g(x)=\ln x$ $g'(x)=\frac{1}{x}\neq 0$

$\exists\xi\in(a,b)$. $\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)} \Rightarrow \frac{f(b)-f(a)}{\ln b-\ln a}=\frac{f'(\xi)}{\frac{1}{\xi}}$

2. $0<a<b$. 证：$\exists\xi\in(a,b)$. 使 $ae^b-be^a=(1-\xi)(a-b).e^\xi$

分析：$\frac{ae^b-be^a}{a-b}=(1-\xi)e^\xi$

$$\parallel$$

$$\frac{\frac{e^b}{b}-\frac{e^a}{a}}{\frac{1}{b}-\frac{1}{a}}$$

证：令$f(x)=\frac{e^x}{x}$ $g(x)=\frac{1}{x}$ $g'(x)=-\frac{1}{x^2}\neq 0$ —柯西$\neq 0$

$\exists\xi\in(a,b)$

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$

Case2．ξ与a,b不可分

方法：$\begin{cases} \xi\to x \\ 去分母移项 \end{cases}$ $\Rightarrow$ 式子$=0\Rightarrow(\quad ?\quad)'=0$

3. $f(x),g(x)\in C[a,b]$ (a,b)内可导．$g'(x)\neq 0$ $(a<x<b)$

证$\exists\xi\in(a,b)$，$\frac{f(\xi)-f(a)}{g(b)-g(\xi)}=\frac{f'(\xi)}{g'(\xi)}$

分析：$f(x)g'(x)-f(a)g'(x)-f'(x)g(b)+f'(x)g(x)=0$

$$[f(x)g(x)-f(a)g(x)-f(x)g(b)]'=0$$

证：令$\varphi(x)=f(x)g(x)-f(a)g(x)-f(x)g(b)$

$\varphi(a)=-f(a)g(b)$，$\varphi(b)=-f(a)g(b)$

$\because \varphi(a)=\varphi(b)$ $\therefore \exists \xi \in (a,b)$, $\varphi'(\xi)=0$

型四. 双中值.

Case 1. 仅有 $f'(\xi)$, $f'(\eta)$

$\begin{cases} \text{找三点} \\ \text{2次 Lg} \end{cases}$

例1. $f(x)\in C[0,1]$, $(0,1)$内可导, $f(0)=0$, $f(1)=1$

—— 无参数不用构造辅助函数

证: ① $\exists c\in(0,1)$, $f(c)=\frac{1}{2}$

② $\exists \xi,\eta \in(0,1)$. $\frac{1}{f'(\xi)}+\frac{1}{f'(\eta)}=2$

证: ① 令 $h(x)=f(x)-\frac{1}{2}$

$h(0)=-\frac{1}{2}$ $h(1)=\frac{1}{2}$

$\because h(0)h(1)<0$. $\therefore \exists c\in(0,1)$, $h(c)=0$

$\Rightarrow f(c)=\frac{1}{2}$

② $\exists \xi\in(0,c)$, $\eta\in(c,1)$ 使

$f'(\xi)=\frac{f(c)-f(0)}{c-0}=\frac{1}{2c}$

$f'(\eta)=\frac{f(1)-f(c)}{1-c}=\frac{1}{2(1-c)}$

$\Rightarrow \frac{1}{f'(\xi)}=2c$, $\frac{1}{f'(\eta)}=2(1-c)$

2. $f(x)\in C[0,1]$, $(0,1)$内可导, $f(0)=0$. $f(1)=1$

① 证: $\exists c\in(0,1)$, $f(c)=1-c$

② $\exists \xi,\eta\in(0,1)$, $f'(\xi)\cdot f'(\eta)=1$

证: ① $h(x)=f(x)-1+x$

$h(0)=-1<0$, $h(1)=1>0$

$\exists c\in(0,1)$. $h(c)=0$

$\Rightarrow f(c)=1-c$

② $\exists \xi \in (0,c)$，$\eta \in (c,1)$.

$$f'(\xi)=\frac{f(c)-f(0)}{c-0}=\frac{1-c}{c} \quad f'(\eta)=\frac{f(1)-f(c)}{1-c}=\frac{c}{1-c}$$

Case2. ξ, η 复杂度不同

方法：留复杂
柯西/拉格朗日

$$\begin{cases} (\quad)' - L_g \\ \dfrac{(\quad)'}{(\quad)'} - C \end{cases}$$

1. $f(x)\in C[a,b]$，(a,b)内可导，$(a>0)$.

证：$\exists \xi, \eta \in (a,b)$. $f'(\xi)=(a+b)\cdot\frac{f'(\eta)}{2\eta}$

分析：$\frac{f'(\eta)}{2\eta}$ — $f(x)$ / x^2

证：令 $g(x)=x^2$ $\quad g'(x)=2x\neq 0$

$\exists \eta \in (a,b)$，使 $\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\eta)}{g'(\eta)}$

$$\Rightarrow \frac{f(b)-f(a)}{b^2-a^2}=\frac{f'(\eta)}{2\eta}$$

$$\frac{f(b)-f(a)}{b-a}=(a+b)\frac{f'(\eta)}{2\eta}$$

$\exists \xi \in (a,b)$，$f'(\xi)=\frac{f(b)-f(a)}{b-a}$

2. $f(x)\in C[a,b]$. (a,b)内可导. $(a>0)$.

证：$\exists \xi, \eta \in (a,b)$. $\quad abf'(\xi)=\eta^2 f'(\eta)$

分析：$\eta^2 f'(\eta) \Rightarrow \frac{f'(\eta)}{\frac{1}{\eta^2}}$ — $f(x)$ / $-\frac{1}{x}$

证：令 $g(x)=-\frac{1}{x}$ $\quad g'(x)=\frac{1}{x^2}\neq 0$.

$\exists \eta \in (a,b)$，使 $\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\eta)}{g'(\eta)}$

$$\frac{f(b)-f(a)}{-\frac{1}{b}+\frac{1}{a}}=\frac{f'(\eta)}{\frac{1}{\eta^2}}$$

$$\Rightarrow ab\,\frac{f(b)-f(a)}{b-a}=\eta^2 f(\eta)$$

型五 $\mathcal{L}_g$ 使用习惯

Case 1. $f(b)-f(a)$ — $\mathcal{L}_g$

1. $f''>0$　$f'(0)$, $f'(1)$, $f(1)-f(0)$ 大小?

解: 1° $f(1)-f(0)=f'(c)(1-0)=f'(c)$　$(0<c<1)$

2° $f''>0 \Rightarrow f'\uparrow$

$\because 0<c<1$

$\therefore f'(0)<\underset{f(1)-f(0)}{\underset{\shortparallel}{f'(c)}}<f'(1)$

2. $\lim\limits_{x\to\infty} f'(x)=e$　$\lim\limits_{x\to\infty}[f(x+1)-f(x)]=e^{2a}$.　$a=?$

解: $f(x+1)-f(x)=f'(\xi)$　$(x<\xi<x+1)$

$\lim\limits_{x\to\infty}[f(x+1)-f(x)]=\lim\limits_{x\to\infty}f'(\xi)=e$

$\Rightarrow e=e^{2a}$　$a=\frac{1}{2}$

3. $\lim\limits_{n\to\infty} n^2(\arctan\frac{1}{n}-\arctan\frac{1}{n+1})$

解: $f(x)=\arctan x$, $f'(x)=\frac{1}{1+x^2}$

$\arctan\frac{1}{n}-\arctan\frac{1}{n+1}=f(\frac{1}{n})-f(\frac{1}{n+1})$

$=f'(\xi)(\frac{1}{n}-\frac{1}{n+1})=\frac{1}{n(n+1)}\cdot\frac{1}{1+\xi^2}$　$(\frac{1}{n+1}<\xi<\frac{1}{n})$

原式 $=\lim\limits_{n\to\infty}\frac{n}{n+1}\cdot\frac{1}{1+\xi^2}=1$

Case 2. $f(a)$, $f(c)$, $f(b)$ — 2次 $\mathcal{L}_g$

1. $f(x)\in C[a,b]$, (a,b)内可导, $|f'(x)|\le M$

$f(x)$在(a,b)内至少一个零点, 证 $|f(a)|+|f(b)|\le M(b-a)$

1° $\exists c\in(a,b)$, $f(c)=0$

2° $\begin{cases} \cancel{f(c)}-f(a)=f'(\xi_1)(c-a) & (a<\xi_1<c) \\ f(b)-\cancel{f(c)}=f'(\xi_2)(b-c) & (c<\xi_2<b) \end{cases}$

3° $|f(a)|\le M(c-a)$

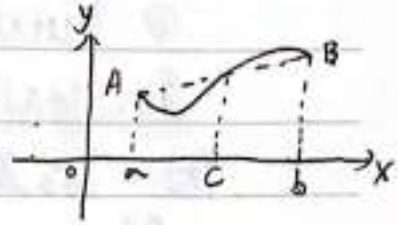
$|f(b)|\le M(b-c)$

2. $f(x) \in C[a,b]$，(a,b) 内二阶可导

L: $y=f(x)$ 连 $A(a,f(a))$，$B(b,f(b))$ 直线

交 L 于 $C(c,f(c))$ $(a<c<b)$

证：$\exists \xi \in (a,b)$，$f''(\xi)=0$



证：

1° $\exists \xi_1 \in (a,c)$，$\xi_2 \in (c,b)$.

$f'(\xi_1) = \dfrac{f(c)-f(a)}{c-a}$，$f'(\xi_2) = \dfrac{f(b)-f(c)}{b-c}$

2° $\because f'(\xi_1) = f'(\xi_2)$

$\therefore \exists \xi \in (\xi_1,\xi_2) \subset (a,b)$

$f''(\xi)=0$

Th 4

Q $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3} \neq \lim\limits_{x\to 0} \dfrac{x-x}{x^3} = 0$

If $\sin x = x + ?\,x^3 + o(x^3)$ 奇函数无偶次项

Th 4 (Taylor) 条件：$f(x)$ 在 $x=x_0$ 邻域内 $(n+1)$ 阶可导

结论：$f(x) = \underset{主}{P_n(x)} + \underset{余}{R_n(x)}$

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$R_n(x) = \begin{cases} \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} & \text{— Lagrange} \\ o\left((x-x_0)^n\right) & \text{— P 皮亚诺型} \end{cases}$$

若 $x_0=0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n)$$

记：

① $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots + \dfrac{x^n}{n!} + o(x^n)$

② $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

③ $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

④ $\frac{1}{1-x}=1+x+x^2+x^3+\cdots+x^n+o(x^n)$

⑤ $\frac{1}{1+x}=1-x+x^2-x^3+\cdots+(-1)^n x^n+o(x^n)$

⑥ $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

⑦ $(1+x)^a=1+ax+\frac{a(a-1)}{2!}x^2+\frac{a(a-1)(a-2)}{3!}x^3$

例1. $\lim\limits_{x\to0}\frac{x-\sin x}{x^3}$

解：$\sin x=x-\frac{x^3}{6}+o(x^3)$

$x-\sin x=\frac{1}{6}x^3+o(x^3)$

原式 $=\lim\limits_{x\to0}\frac{\frac{1}{6}x^3+o(x^3)}{x^3}=\frac{1}{6}$

2. $\lim\limits_{x\to0}\frac{e^{-\frac{x^2}{2}}-1+\frac{x^2}{2}}{x^3\sin x}$

$=\lim\limits_{x\to0}\frac{e^{-\frac{x^2}{2}}-1+\frac{x^2}{2}}{x^4}$

$\because e^x=1+x+\frac{x^2}{2!}+o(x^2)$

$e^{-\frac{x^2}{2}}=1-\frac{x^2}{2}+\frac{1}{8}x^4+o(x^4)$

$\therefore e^{-\frac{x^2}{2}}-1+\frac{x^2}{2}=\frac{1}{8}x^4+o(x^4)\sim\frac{1}{8}x^4$

$\therefore$ 原式 $=\frac{1}{8}$

3. $\lim\limits_{x\to0}\frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}$

解：$(1+x)^a=1+ax+\frac{a(a-1)}{2!}x^2+o(x^2)$

$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+o(x^2)$

$\sqrt{1-x}=1-\frac{1}{2}x-\frac{1}{8}x^2+o(x^2)$

$\sqrt{1+x}+\sqrt{1-x}-2\sim-\frac{1}{4}x^2$ $\quad\therefore$ 原式 $=-\frac{1}{4}$

☆ Part II 单调性与极值

$y=f(x)$：

1° $x\in D$；

2° $f'(x)\begin{cases}=0\\ 无\end{cases}$ (不一定)；

3° 判别法：

方法一：（第一充分条件）

Th1. ① $\begin{cases} x<x_0, & f'(x)<0 \\ x>x_0, & f'(x)>0 \end{cases} \Rightarrow x=x_0$ 为极小点

② $\begin{cases} x<x_0, & f'(x)>0 \\ x>x_0, & f'(x)<0 \end{cases} \Rightarrow x=x_0$ 为极大点

方法二：（第二充分条件）

Th2. $f'(x_0)=0$. $f''(x_0)\begin{cases} >0, & \text{极小点} \\ <0, & \text{极大点} \end{cases}$

证：设 $f'(x_0)=0$, $f''(x_0)>0$

$f''(x_0)=\lim\limits_{x\to x_0}\frac{f'(x)-\cancel{x}}{x-x_0}>0$

$\exists\delta>0$, 当 $0<|x-x_0|<\delta$ 时 $\frac{f'(x)}{x-x_0}>0$　　保号性

$\begin{cases} f'(x)<0, & x\in(x_0-\delta,x_0) \\ f'(x)>0, & x\in(x_0,x_0+\delta) \end{cases}$

$\Rightarrow x=x_0$ 为极小点

设 $f'(x_0)=0$ $f''(x_0)<0$

$f''(x_0)=\lim\limits_{x\to x_0}\frac{f'(x)}{x-x_0}<0$

$\exists\delta>0$, 当 $0<|x-x_0|<\delta$ 时 $\frac{f'(x)}{x-x_0}<0$

$\begin{cases} f'(x)>0 & x\in(x_0-\delta,x_0) \\ f'(x)<0 & x\in(x_0,x_0+\delta) \end{cases}$

$\Rightarrow x=x_0$ 为极大点

题型一：不等式证明

1. $e<a<b$. 证：$a^b>b^a$

证：$a^b>b^a \Leftrightarrow b\ln a-a\ln b>0$

令 $f(x)=x\ln a-a\ln x$, $\underline{f(a)=0}$

$f'(x)=\ln a-\frac{a}{x}>0$　$(x>a)$

$$\begin{cases} f(a)=0 \\ f'(x)>0 \quad (x>a) \end{cases} \Rightarrow f(x)>0 \quad (x>a)$$

$\because b>a \qquad \therefore f(b)>0$

2. $x>0$，证：$\frac{x}{1+x}<\ln(1+x)<x$

证：令 $f(x)=\ln(1+x)-\frac{x}{1+x}$，$f(0)=0$

$f'(x)=\frac{1}{1+x}-\frac{1}{(1+x)^2}>0 \qquad (x>0)$

$$\begin{cases} f(0)=0 \\ f'(x)>0 \quad (x>0) \end{cases} \Rightarrow f(x)>0 \quad (x>0)$$

即 $x>0$ 时，$\frac{x}{1+x}<\ln(1+x)$

令 $g(x)=x-\ln(1+x)$，$g(0)=0$

$g'(x)=1-\frac{1}{1+x}>0 \quad (x>0)$

$$\begin{cases} g(0)=0 \\ g'(x)>0 \quad (x>0) \end{cases} \Rightarrow g(x)>0 \quad (x>0)$$

即 $x>0$ 时 $\ln(1+x)<x$

3. $f(a)=g(a)$，$f'(a)=g'(a)$，$f''(x)>g''(x) \quad (x>a)$

证 $x>a$ 时 $f(x)>g(x)$.

证：$\varphi(x)=f(x)-g(x)$

$\varphi(a)=0$，$\varphi'(a)=0 \qquad \varphi''(x)>0 \quad (x>a)$

$$\begin{cases} \varphi'(a)=0 \\ \varphi''(x)>0 \quad (x>a) \end{cases} \Rightarrow \varphi'(x)>0 \quad (x>a)$$

$$\begin{cases} \varphi(a)=0 \\ \varphi'(x)>0 \quad (x>a) \end{cases} \Rightarrow \varphi(x)>0 \quad (x>a)$$

4. $0<a<b$，证：$\ln b-\ln a>\frac{2(b-a)}{a+b}$

证：$\ln b-\ln a>\frac{2(b-a)}{a+b}$

$\Leftrightarrow \quad (a+b)(\ln b-\ln a)-2(b-a)>0$

令 $f(x)=(a+x)(\ln x-\ln a)-2(x-a)$，$f(a)=0$

$f'(x) = \ln x - \ln a + \frac{a+x}{x} - 2$

$= \ln x - \ln a + \frac{a}{x} - 1$ $f'(a) = 0$

$f''(x) = \frac{1}{x} - \frac{a}{x^2} = \frac{x-a}{x^2} > 0$ $(x>a)$

$\begin{cases} f'(a) = 0 \\ f''(x) > 0 \quad (x>a) \end{cases} \Rightarrow f'(x) > 0 \quad (x>a)$

$\begin{cases} f(a) = 0 \\ f'(x) > 0 \quad (x>a) \end{cases} \Rightarrow f(x) > 0 \quad (x>a)$

$\because b > a$

$\therefore f(b) > 0$.

型二 方程根讨论

① 零点定理

如：证 $x^2 - 3x + 1 = 0$ 至少一个正根

$f(x) = x^2 - 3x + 1$

$f(0) = 1$ $f(1) = -1$ $f(0)f(1) < 0$.

$\exists c \in (0,1)$, $f(c) = 0$

② (Rolle) $f(x) \Rightarrow F(x)$ 即 $F'(x) = f(x)$

If $F(a) = F(b)$

then. $\exists c \in (a,b)$, $F'(c) = 0 \Rightarrow f(c) = 0$

例1. $a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0$

证 $a_0 + a_1 x + a_2 x^2 = 0$ 至少一个正根.

证：令 $f(x) = a_0 + a_1 x + a_2 x^2$

$F(x) = a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3$

$F(0) = F(1) = 0$

$\exists c \in (0,1)$. 使 $F'(c) = 0 \Rightarrow f(c) = 0$.

③(草图法)

1° $f(x)\quad (x\in D)$.

2° $f'(x)\begin{cases}=0 \Rightarrow \text{极值};\\ \text{无}\end{cases}$

3° 关注两侧，作草图.

1. 讨论 $\ln x=\frac{x}{e}-2$ 几根?

解. 1° $f(x)=\ln x-\frac{x}{e}+2\quad (x>0)$

2° $f'(x)=\frac{1}{x}-\frac{1}{e}=0 \Rightarrow x=e$

$f''(x)=-\frac{1}{x^2}\quad \because f''(e)<0$

$\therefore x=e$ 为最大点

$M=f(e)=2>0$

3° $f(0+0)=-\infty$

$\lim\limits_{x\to+\infty}f(x)=-\infty$

$\therefore f(x)$ 2个零点

$\therefore$ 原方程两个根

2. $a>0$. 讨论 $xe^{-x}=a$ 几根

解: 1° $f(x)=xe^{-x}-a\quad (x>0)$

2° $f'(x)=e^{-x}-xe^{-x}=(1-x)e^{-x}=0 \Rightarrow x=1$

$\begin{cases}f'(x)>0, & 0<x<1\\ f'(x)<0, & x>1\end{cases} \Rightarrow x=1$ 为最大点.

$M=f(1)=\frac{1}{e}-a$.

3° ① $M<0$. 即 $a>\frac{1}{e}$ 方程无根

② $M=0$, 即 $a=\frac{1}{e}$ 方程有一根 $x=1$.

③ $M>0$. 即 $0<a<\frac{1}{e}$

$f(0)=-a<0\qquad \lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to+\infty}\left(\frac{x}{e^x}-a\right)=-a<0$

方程两根.

型三 极值点判断

1° $x\in D$;

2° $f'(x)\begin{cases}=0\\ 无\end{cases}$

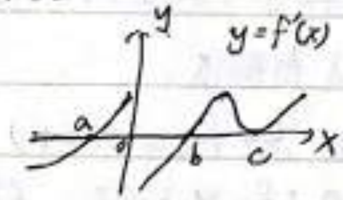
3° 判别法

1. $f'(1)=0$. $\lim\limits_{x\to 1}\dfrac{f'(x)}{\sin\pi x}=-1$, $x=1$?

解：$\exists\,\delta>0$ 当 $0<|x-1|<\delta$ 时 $\dfrac{f'(x)}{\sin\pi x}<0$

$$\begin{cases} f'(x)<0, & x\in(1-\delta,1)\\ f'(x)>0, & x\in(1,1+\delta)\end{cases}\Rightarrow x=1 \text{ 为极小点}$$

2. $f(x)\in C(-\infty,+\infty)$.



解：1° $x\in(-\infty,+\infty)$

2° $f'(x)\begin{cases}=0\\ 无\end{cases}\Rightarrow x=a,0,b,c$

3° $\begin{cases} x<a, & f'(x)<0\\ x>a, & f'(x)>0\end{cases}\Rightarrow x=a$ 极小点.

$\begin{cases} x<0, & f'(x)>0\\ x>0, & f'(x)<0\end{cases}\Rightarrow x=0$ 为极大值点

$\begin{cases} x<b, & f'<0\\ x>b, & f'>0\end{cases}\Rightarrow x=b$ 为极小点

$\begin{cases} x<c & f'>0\\ x>c & f'>0\end{cases}\Rightarrow x=c$ 不是极值点.

3. $f(x)$, $xf''(x)-3xf'^2(x)=1-e^{-2x}$

$a>0$, 且 $x=a$ 为极值点, 问极大或极小?

解: 1° $f'(a)=0$

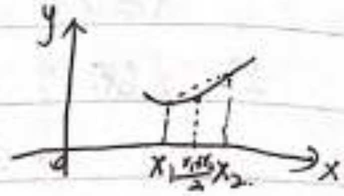
2° $af''(a)=1-e^{-2a}\Rightarrow f''(a)=\dfrac{1-e^{-2a}}{a}>0$

$x=a$ 为极小点.

Part Ⅲ 零碎问题

一、凹凸性.

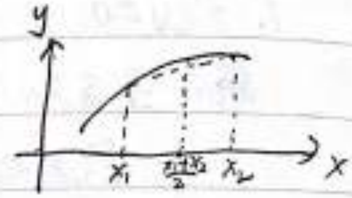
(一) def - $y=f(x)$ ($x\in D$).



1. If $\forall x_1,x_2\in D$ 且 $x_1\neq x_2$，有

$$f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}$$

称 $y=f(x)$ 在D内 凹的.



2. If $\forall x_1,x_2\in D$ 且 $x_1\neq x_2$

$$f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}$$

称 $y=f(x)$ 在D内为凸函数.

(二) 判别法.

Th. $y=f(x)$ ($x\in D$)

① If $\forall x\in I$，$f''>0$，则 $y=f(x)$ 在I内凹的

② If $\forall x\in I$，$f''<0$，则 $y=f(x)$ 在I内凸的

如：1. $y=x^3$.　令 $y''=6x=0 \Rightarrow x=0$

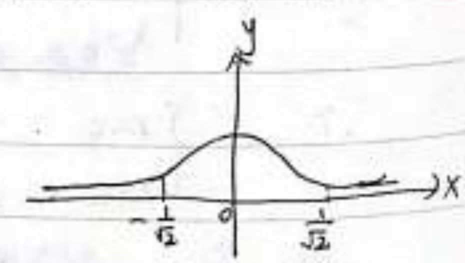
当 $x\in(-\infty,0)$时，$y''<0$.　$y=x^3$ 在 $(-\infty,0)$内凸的

当 $x\in(0,+\infty)$时，$y''>0$.　$y=x^3$ 在 $(0,+\infty)$内凹的；

2. $y=e^{-x^2}$

解：$y'=-2xe^{-x^2}$　$y''=-2e^{-x^2}+4x^2e^{-x^2}=4(x^2-\frac{1}{2})e^{-x^2}$

令 $y''=0 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$



当 $x\in(-\infty,-\frac{1}{\sqrt{2}})$ 时，$y''>0$

当 $x\in(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ 时，$y''<0$

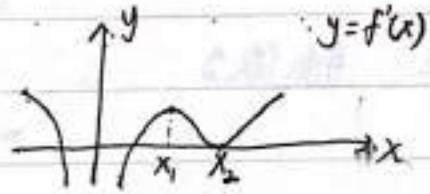
当 $x\in(\frac{1}{\sqrt{2}},+\infty)$ 时，$y''>0$

$y=e^{-x^2}$ 在 $(-\infty,-\frac{1}{\sqrt{2}})$ 内凹；

$y=e^{-x^2}$ 在 $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ 内凸；

$y=e^{-x^2}$ 在 $(\frac{1}{\sqrt{2}},+\infty)$ 内凹.

注：$(-\frac{1}{\sqrt{2}},e^{-\frac{1}{2}})$、$(\frac{1}{\sqrt{2}},e^{-\frac{1}{2}})$ 称为拐点。



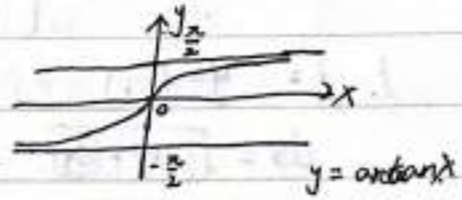
3. $f(x) \in C(-\infty,+\infty)$

解：1°. $x \in (-\infty,+\infty)$

2°. $f''(x) \begin{cases} =0 \\ 无 \end{cases} \Rightarrow x=0, x_1, x_2.$

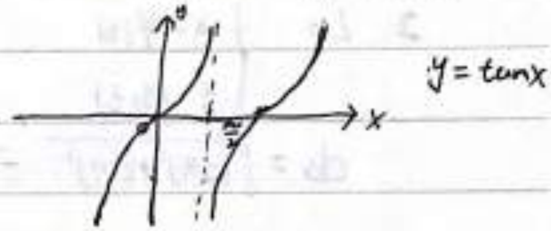
3° $\begin{cases} x<0, & f''<0 \\ x>0, & f''>0 \end{cases} \Rightarrow (0, f(0))$ 为拐点

二、渐近线



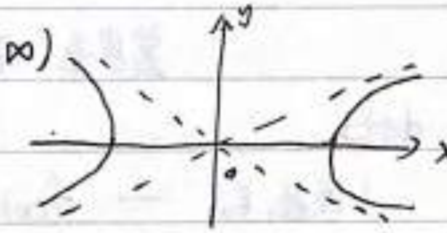
1. 水平渐近线 — If $\lim\limits_{x\to\infty} f(x) = A$

$y=A$ 为 $y=f(x)$ 的水平渐近线



2. 铅直渐近线 — If $\begin{cases} f(a-0)=\infty \\ f(a+0)=\infty \\ \lim\limits_{x\to a} f(x)=\infty \end{cases}$

称 $x=a$ 为 $y=f(x)$ 的铅直渐近线



3. 斜渐近线 — If $\lim\limits_{x\to\infty} \frac{f(x)}{x} = a \ (\neq 0, \infty)$

$\lim\limits_{x\to\infty} [f(x)-ax] = b.$

$y=ax+b$ 为斜渐近线

例1. $y=\frac{2x^2-x-1}{x^2-1}$

解. $\lim\limits_{x\to\infty} y = 2$. $\Rightarrow y=2$ 为水平渐近线；

$\lim\limits_{x\to -1} y = \infty$ ，$x=-1$ 为铅直渐近线.

$\lim\limits_{x\to 1} y = \lim\limits_{x\to 1} \frac{4x-1}{2x} = \frac{3}{2}$ ，$x=1$ 不是铅直渐近线

2. $y=\frac{x^2-3x+2}{x-1}$

解：$\lim\limits_{x\to\infty} y = \infty$ $\Rightarrow$ 无水平渐近线；

$\lim\limits_{x\to 1} y = \lim\limits_{x\to 1} (x-2) = -1 \Rightarrow x=1$ 不是铅直渐近线.

$\lim\limits_{x\to\infty} \frac{y}{x} = 1$

$\lim\limits_{x\to\infty} (y-x) = \lim\limits_{x\to\infty} \left(\frac{x^2-3x+2}{x-1} - x\right) = \lim\limits_{x\to\infty} \frac{-2x+2}{x-1} = -2$

$\therefore y=x-2$ 为斜渐近线.

曲率、曲率半径互为倒数
K　　ρ

曲率：$K=\dfrac{|y''|}{[1+(y')^2]^{\frac{3}{2}}}$

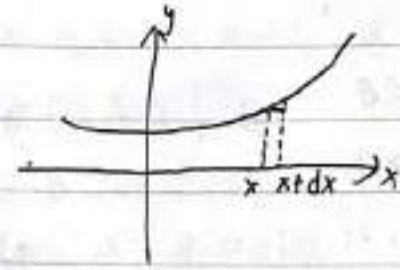
$\rho=\dfrac{1}{K}$

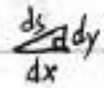
$\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$

$K=\dfrac{|\varphi'(t)\psi''(t)-\varphi''(t)\psi'(t)|}{[\varphi'^2(t)+\psi'^2(t)]^{\frac{3}{2}}}$

三、弧微分

$(\Delta s)^2\approx(\Delta x)^2+(\Delta y)^2$





$(ds)^2=(dx)^2+(dy)^2$

1. L: $y=f(x)$

$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{1+(\frac{dy}{dx})^2}\,dx=\sqrt{1+f'^2(x)}\,dx$；

2. L: $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$

$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt=\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$

第四章 不定积分

一、defs.

1.原函数 — $f(x)$, $F(x)$, If $F'(x)=f(x)$

$F(x)$ 称为$f(x)$的原函数.

Note:

① 连续函数一定存在原函数；

② 若$f(x)$ $\exists$原函数，则$\exists$无数个原函数，且任两个原函数相差常数

2. 不定积分 — $\int f(x)dx=F(x)+C$

二、不定积分工具.

(一) 基本公式.

1. $\int k\,dx=kx+C$

2. $\int x^a dx=\begin{cases} \frac{1}{a+1}x^{a+1}+C, & a\neq-1 \\ \ln|x|+C & a=-1 \end{cases}$

3. ① $\int a^x dx = \frac{a^x}{\ln a} + C$

② $\int e^x dx = e^x + C$

4. ① $\int \sin x \, dx = -\cos x + C$

② $\int \cos x \, dx = \sin x + C$

③ $\int \tan x \, dx = -\ln|\cos x| + C$

④ $\int \cot x \, dx = \ln|\sin x| + C$

⑤ $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

⑥ $\int \csc x \, dx = \ln|\csc x - \cot x| + C$

⑦ $\int \sec^2 x \, dx = \tan x + C$

⑧ $\int \csc^2 x \, dx = -\cot x + C$

⑨ $\int \sec x \tan x \, dx = \sec x + C$

⑩ $\int \csc x \cot x \, dx = -\csc x + C$

5、平方和、差:

① $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

② $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C$

③ $\int \frac{1}{1+x^2} dx = \arctan x + C$

④ $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$

⑤ $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln(x+\sqrt{x^2+a^2}) + C$

⑥ $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln|x+\sqrt{x^2-a^2}| + C$

⑦ $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$

⑧ $\int \sqrt{a^2-x^2} \, dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x\sqrt{a^2-x^2} + C$

(二) 积分法

1、换元积分法

Case 1 第一类换元积分法.

例1. $\int \frac{e^x}{4+e^{2x}} dx = \int \frac{1}{2^2+(e^x)^2} d(e^x)$

$\xlongequal{e^x=t} \int \frac{1}{2^2+t^2} dt = \frac{1}{2} \arctan \frac{t}{2} + C = \frac{1}{2} \arctan \frac{e^x}{2} + C$

例2. $\int \frac{x}{(2x+1)^2}dx = \frac{1}{4}\int \frac{(2x+1)-1}{(2x+1)^2}d(2x+1)$

$\xlongequal{2x+1=t} \frac{1}{4}\int \frac{t-1}{t^2}dt = \frac{1}{4}\int (\frac{1}{t}-\frac{1}{t^2})dt$

$= \frac{1}{4}(\ln|t| + \frac{1}{t}) + C = \frac{1}{4}[\ln|2x+1| + \frac{1}{2x+1}] + C$

3. $\int (1-\frac{1}{x^2})\cos(x+\frac{1}{x})dx$

$= \int \cos(x+\frac{1}{x})d(x+\frac{1}{x})$

$= \sin(x+\frac{1}{x}) + C$

4. $\int \frac{1}{x\ln^2 x}dx = \int \frac{1}{\ln^2 x}d(\ln x) = -\frac{1}{\ln x} + C$

5. $\int \frac{dx}{\sqrt{x(1-x)}} = 2\int \frac{dx}{2\sqrt{x}\cdot\sqrt{1-x}} = 2\int \frac{d\sqrt{x}}{\sqrt{1-(\sqrt{x})^2}} = 2\arcsin\sqrt{x} + C$

6. $\int \tan x\,dx = -\int \frac{1}{\cos x}d(\cos x) = -\ln|\cos x| + C$

7. $\int \cot x\,dx = \int \frac{1}{\sin x}d(\sin x) = \ln|\sin x| + C$

8. $\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}}d(\frac{x}{a}) = \arcsin\frac{x}{a} + C$

9. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2}d(\frac{x}{a}) = \frac{1}{a}\arctan\frac{x}{a} + C$

10. $\int \frac{\arcsin^2 x}{\sqrt{1-x^2}}dx = \int \arcsin^2 x\,d(\arcsin x) = \frac{1}{3}\arcsin^3 x + C$

第一类… $\int f[\varphi(x)]\varphi'(x)dx = \int f[\varphi(x)]d\varphi(x) \xlongequal{\varphi(x)=t} \int f(t)dt$

$= F(t) + C = F[\varphi(x)] + C$

例11. $\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int \frac{(2x+1)+1}{x^2+x+1}dx = \frac{1}{2}[\int \frac{d(x^2+x+1)}{x^2+x+1} + \int \frac{d(x+\frac{1}{2})}{(\frac{\sqrt{3}}{2})^2+(x+\frac{1}{2})^2}]$

$= \frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\times\frac{2}{\sqrt{3}}\arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$

12. $\int \frac{\sin 2x}{1+\sin^4 x}dx$

解：$\because (\sin^2 x)' = 2\sin x\cos x = \sin 2x$

$\therefore \int \frac{\sin 2x}{1+\sin^4 x}dx = \int \frac{d(\sin^2 x)}{1+(\sin^2 x)^2} = \arctan\sin^2 x + C$

Case 2. 第二类换元积分法.

① 无理 $\Rightarrow$ 有理 (不一定)

例1. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin\sqrt{x}\,d(\sqrt{x}) = -2\cos\sqrt{x} + C$

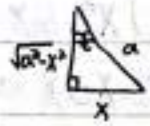
例2. $\int \frac{dx}{\sqrt{x}(4+x)} = 2\int \frac{d\sqrt{x}}{2^2+(\sqrt{x})^2} = \arctan\frac{\sqrt{x}}{2} + C$

例3. $\int \frac{dx}{1+\sqrt{x}} \xlongequal{x=t^2} 2\int \frac{t}{1+t}dt = 2\int (1-\frac{1}{1+t})dt$

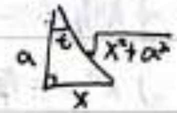
$= 2(t-\ln|t+1|) + C = 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$

② 平方和、差：

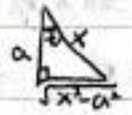
$\sqrt{a^2-x^2} \xlongequal{x=a\sin t} a\cos t$



$\sqrt{x^2+a^2} \xlongequal{x=a\tan t} a\sec t$

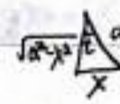


$\sqrt{x^2-a^2} \xlongequal{x=a\sec t} a\tan t$



例1. $\int\sqrt{a^2-x^2}\,dx \xlongequal{x=a\sin t} \int a^2\cos^2 t\,dt = \frac{a^2}{2}\int(1+\cos 2t)\,dt$

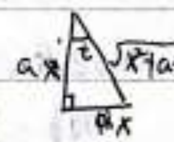
$= \frac{a^2}{2}(t+\frac{1}{2}\sin 2t)+C = \frac{a^2}{2}t+\frac{a^2}{2}\sin t\cos t+C$



$= \frac{a^2}{2}\arcsin\frac{x}{a}+\frac{a^2}{2}\times\frac{x}{a}\times\frac{\sqrt{a^2-x^2}}{a}+C$

$= \frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$

2. $\int\frac{dx}{\sqrt{x^2+a^2}} \xlongequal{x=a\tan t} \int\frac{a\sec^2 t}{a\sec t}dt = \int\sec t\,dt$

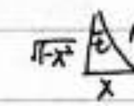


$= \ln|\sec t+\tan t|+C$

$= \ln|\frac{x}{a}+\frac{\sqrt{x^2+a^2}}{a}|+C$

$= \ln(x+\sqrt{x^2+a^2})+C$

3. $\int\frac{dx}{x^2\sqrt{1-x^2}} \xlongequal{x=\sin t} \int\frac{\cos t\,dt}{\sin^2 t\cos t} = \int\csc^2 t\,dt$



$= -\cot t+C$

$= -\frac{\sqrt{1-x^2}}{x}+C$

2.分部积分法.

$(uv)' = u'v+uv' \Rightarrow uv = \int v\,du+\int u\,dv$

$\Longrightarrow \underline{\int u\,dv = uv-\int v\,du}$

Case 1. $\int x^n\cdot e^x\,dx$

例1. $\int x^2e^x\,dx = \int x^2\,d(e^x) = x^2e^x-\int e^x\,d(x^2)$

$= x^2e^x-2\int xe^x\,dx = x^2e^x-2\int x\,d(e^x) = x^2e^x-2(xe^x-\int e^x\,dx)$

$= x^2e^x-2xe^x+2e^x+C$

Case 2. $\int x^n \cdot \ln x \, dx$

例2. $\int x^2 \ln x \, dx = \int \ln x \, d(\frac{1}{3}x^3) = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \cdot d\ln x$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

Case 3. $\int x^n \times$ 三角 dx 　　sin, cos 三角 降级一次　三角到后面　tan, cot, sec, csc 二次

例3. $\int x \sin^2 x \, dx = \frac{1}{2}\int x(1-\cos 2x)dx = \frac{1}{2}\int (x - x\cos 2x)dx$

$= \frac{1}{4}x^2 - \frac{1}{4}\int x \, d\sin 2x = \frac{1}{4}x^2 - \frac{1}{4}(x\sin 2x - \int \sin 2x \, dx)$

$= \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

例4. $\int x \sec^2 x \, dx = \int x \, d(\tan x) = x\tan x - \int \tan x \, dx$

$= x\tan x + \ln|\cos x| + C$

例5. $\int x \tan x \sec x \, dx = \int x \, d(\sec x) = x\sec x - \int \sec x \, dx$

$= x\sec x - \ln|\sec x + \tan x| + C$

Case 4. $\int x^n \times$ 反三角 dx

例6. $\int x^2 \arctan x \, dx = \int \arctan x \, d(\frac{1}{3}x^3)$

$= \frac{1}{3}x^3 \arctan x - \int \frac{1}{3}x^3 d(\arctan x)$

$= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{(x^3+x)-x}{1+x^2} dx$

$= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int (x - \frac{x}{1+x^2}) dx$

$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\int \frac{d(1+x^2)}{1+x^2}$

$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$

例7. $\int \arcsin x \, dx$

$= x\arcsin x + \int \frac{-2x}{2\sqrt{1-x^2}} dx$

$= x\arcsin x + \int \frac{d(1-x^2)}{2\sqrt{1-x^2}}$

$= x\arcsin x + \sqrt{1-x^2}$

Case 5. $\int e^{ax} \times \begin{cases} \cos bx \\ \sin bx \end{cases} dx$ 　　哪个到后都可

例8. $\int e^{2x}\cos x\,dx$

解：$I=\int e^{2x}\cos x\,dx = \int e^{2x}\,d(\sin x) = e^{2x}\sin x - 2\int e^{2x}\sin x\,dx$

$= e^{2x}\sin x + 2\int e^{2x}\,d(\cos x) = e^{2x}\sin x + 2(e^{2x}\cos x - 2\int e^{2x}\cos x\,dx)$

$= e^{2x}\sin x + 2e^{2x}\cos x - 4I$

$I = \frac{1}{5}e^{2x}(\sin x + \cos x) + C$

case 6. $\int \begin{cases} \sec^n x \\ \csc^n x \end{cases} dx$ (n为奇数)

如：$\int \cancel{\sin x} \sec^4 x\,dx = \int (1+\tan^2 x)\,d(\tan x) = \tan x + \frac{1}{3}\tan^3 x + C$

又如：$\int \tan^3 x\sec x\,dx = \int \cancel{\tan x}(\sec^2 x - 1)\,d(\sec x)$

$= \frac{1}{3}\sec^3 x - \sec x + C$

例9. $\int \sec^3 x\,dx$

解：$I = \int \sec x\,d(\tan x) = \sec x\tan x - \int \sec x\tan^2 x\,dx$

$= \sec x\tan x - \int \sec x\cdot(\sec^2 x - 1)\,dx$

$= \sec x\tan x - I + \ln|\sec x + \tan x|$ *

$I = \frac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|) + C$

三.特殊积分类型

(一) $\int R(x)\,dx$. rational 有理的. poly - 多 prefix前缀

$R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$、$Q(x)$ are polynomials. 多项式

If $\deg P < \deg Q$ —真分式

if $\deg P \geq \deg Q$ —假分式

1° If $R(x)$为假分式 $R(x)$=多+真：

如：$\frac{x^3-2x^2-4}{x^2+1} = \frac{(x^3+x)-2x^2-x-4}{x^2+1} = x - \frac{(2x^2+2)+x+2}{x^2+1} = x-2-\frac{x+2}{x^2+1}$

2° If $R(x)$为真分式

$R(x)$分子不变 分母因式分解 ⇒ 拆成部分和.

① $R(x) = \frac{2x+3}{x^2-x-6} = \frac{2x+3}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$

$A(x-3)+B(x+2)=2x+3 \Rightarrow \begin{cases} A+B=2 \\ -3A+2B=3 \end{cases}$

② $R(x)=\frac{3x-1}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$

如: $R(x)=\frac{1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}$

③ $R(x)=\frac{5x^2-x+1}{(x-1)^2(x^2+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Bx+C}{x^2+1}$

例1. ① $\int\frac{dx}{x^2-x-6}=\int\frac{dx}{(x-3)(x+2)}=\frac{1}{5}\int\left(\frac{1}{x-3}-\frac{1}{x+2}\right)dx=\frac{1}{5}\ln\left|\frac{x-3}{x+2}\right|+C$

② $\int\frac{dx}{x^2+2x+5}=\int\frac{1}{2^2+(x+1)^2}d(x+1)=\frac{1}{2}\arctan\frac{x+1}{2}+C$

例2. ① $\int\frac{5x-1}{x^2-x-6}dx$

解: $\frac{5x-1}{x^2-x-2}=\frac{5x-1}{(x-2)(x+1)}=\frac{A}{x+1}+\frac{B}{x-2}$

$A(x-2)+B(x+1)=5x-1 \Rightarrow \begin{cases}A+B=5\\-2A+B=-1\end{cases} \Rightarrow \begin{cases}A=2\\B=3\end{cases}$

$\int\frac{5x-1}{x^2-x-2}dx=2\int\frac{dx}{x+1}+3\int\frac{dx}{x-2}=2\ln|x+1|+3\ln|x-2|+C$

例3. $\int\frac{dx}{x(x-1)^2}$

解: $\frac{1}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$

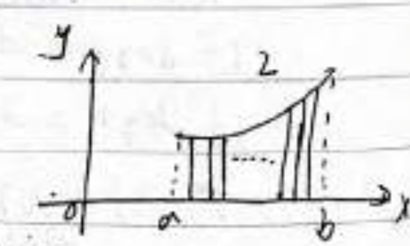
$A(x-1)^2+Bx(x-1)+Cx=1 \quad \begin{cases}A+B=0\\-2A-B+C=0\\A=1\end{cases} \Rightarrow A=1\ \ B=-1\ \ C=1$

原式 $=\int\left(\frac{1}{x}-\frac{1}{x-1}+\frac{1}{(x-1)^2}\right)dx$

$=\ln\left|\frac{x}{x-1}\right|-\frac{1}{x-1}+C$

第四章 定积分

一、背景:

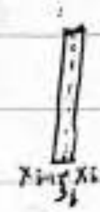


1、$L=\ y=f(x)\geq 0\ \ (a\leq x\leq b)$

1°. $a=x_0<x_1<\cdots<x_n=b$.

$[a,b]=[x_0,x_1]\cup[x_1,x_2]\cup\cdots\cup[x_{n-1},x_n]$

2° $\forall\ \xi_i\in[x_{i-1},x_i]$



$\Delta S_i\approx f(\xi_i)\Delta x_i$

$S\approx\sum_{i=1}^{n}f(\xi_i)\Delta x_i$

3°. $\lambda=\max\{\Delta x_1,\Delta x_2,\cdots,\Delta x_n\}$.

$S=\lim\limits_{\lambda\to0}\sum\limits_{i=1}^{n}f(\xi_i)\Delta x_i$

2. $v=v(t)$, $t\in[a,b]$. 求S.

1° $a=t_0<t_1<t_2<\cdots<t_n=b$.

$[a,b]=[t_0,t_1]\cup[t_1,t_2]\cup[t_2\cdots\cup[t_{n-1},t_n]$

2° $\forall\xi_i\in[t_{i-1},t_i]$

$\Delta S_i\approx v(\xi_i)\Delta t_i$

$S\approx\sum\limits_{i=1}^{n}v(\xi_i)\Delta t_i$;

3°. $\lambda=\max\{\Delta t_1,\Delta t_2,\cdots,\Delta t_n\}$.

$S=\lim\limits_{\lambda\to0}\sum\limits_{i=1}^{n}v(\xi_i)\Delta t_i$

思想: $y=f(x)$. $x\in[a,b]$

1° $a=x_0<x_1<\cdots<x_n=b$.

$[a,b]=[x_0,x_1]\cup\cdots\cup[x_{n-1},x_n]$

2° $\forall\xi_i\in[x_{i-1},x_i]$

$\sum\limits_{i=1}^{n}f(\xi_i)\Delta x_i$;

3° $\lambda=\max\{\Delta x_1,\cdots,\Delta x_n\}$

$\lim\limits_{\lambda\to0}\sum\limits_{i=1}^{n}f(\xi_i)\Delta x_i$

二、定积分定义:

$y=f(x)$, 在 $x\in[a,b]$ 上有界: → 定积分(黎曼积分)

1°. $a=x_0<\cdots<x_n=b$;

2°. $\forall\xi_i\in[x_{i-1},x_i]$

$\sum\limits_{i=1}^{n}f(\xi_i)\Delta x_i$;

3° $\lambda=\max\{\Delta x_1,\cdots,\Delta x_n\}$

若 $\lim\limits_{\lambda\to0}\sum\limits_{i=1}^{n}f(\xi_i)\Delta x_i$ 存在

称 $f(x)$ 在 $[a,b]$ 上可积, 极限值 称为 $f(x)$ 在 $[a,b]$ 上的定积分

记 $\int_a^b f(x)dx$.

$\int_a^b f(x)dx\triangleq\lim\limits_{\lambda\to0}\sum\limits_{i=1}^{n}f(\xi_i)\Delta x_i$.

Notes:

① $\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i$ 与 $\begin{cases}\text{区间分法}\\ \xi_i\text{取法}\end{cases}$ 无关

② f(x)在[a, b]上有界是可积必要条件. ⟶ 定积分(黎曼积分)

反例: $f(x)=\begin{cases}1, & x\in Q\\ -1, & x\in R\backslash Q\end{cases}$ 无理

f(x)在[a, b]上有界.

对 $\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i$.

Case 1. $\forall \xi_i\in Q$ $\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i = b-a$;

case 2. $\forall \xi_i\in R/Q$. $\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i = -(b-a)$

$\therefore \lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i$ 不存在 ⟹ f(x)在[a, b]上不可积.

③ $\lambda\to 0 \not\Leftarrow n\to\infty$

"⟹" $b-a=\Delta x_1+\cdots+\Delta x_n \leq n\lambda$

$n \geq \frac{b-a}{\lambda} \to +\infty \quad (\lambda\to 0)$

"$\not\Leftarrow$" a x_0 x_1 b

$n\to\infty$, 但 $\lambda=\frac{b-a}{2}$

④ 设f(x)在[0, 1]上可积.

1°. $[0,1]=[0,\frac{1}{n}]\cup[\frac{1}{n},\frac{2}{n}]\cup\cdots\cup[\frac{n-1}{n},\frac{n}{n}]$

$\Delta x_1=\Delta x_2=\cdots\Delta x_n=\frac{1}{n}$

$(\lambda\to 0 \Leftrightarrow n\to\infty)$

2°. $\xi_1=\frac{1}{n}$, $\xi_2=\frac{2}{n}$, …… $\xi_i=\frac{i}{n}$, … . $\xi_n=\frac{n}{n}$

$\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i = \frac{1}{n}\sum\limits_{i=1}^{n} f(\frac{i}{n})$;

3° $\int_0^1 f(x)\,dx = \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n} f(\frac{i}{n})$

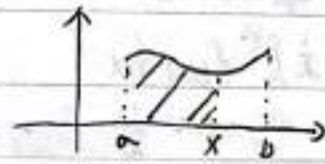
例1. $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right)$ 分子齐.分母齐

解：原式 $= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i^2}} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{(\frac{i}{n})^2+1}} = \int_0^1\frac{1}{\sqrt{x^2+1}}$

$= \cancel{\ln(x+\sqrt{x^2+a^2})} = \ln(x+\sqrt{x^2+1})\Big|_0^1$

$= \ln(1+\sqrt{2})$

三.积分基本公式



$f(x)\in C[a,b]$.

$\int_a^x f(x)\,dx = \int_a^x f(t)\,dt \triangleq \Phi(x)$ — 积分上限函数.

Q1. $\int x^2dx = \frac{x^3}{3}+C \neq \int t^2\,dt = \frac{t^3}{3}+C$

Q2. $\int_0^1 x^2dx = \frac{1}{3} = \int_0^1 t^2\,dt = \frac{1}{3}$

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du = \cdots$

1. 定积分由 $\left\{\begin{array}{l}\text{上、下限}\\ \text{函数关系}\end{array}\right.$ 决定，与积分变量无关.

Q1. $\int_a^x f(x)\,dx$ 表达式 X 与上限 X 不同

Q2. $\int_a^x f(x,t)\,dt$，表达式 X 与上限 X 同

Th1. $f(x)\in C[a,b]$. $\Phi(x)=\int_a^x f(t)\,dt$ 则 $\Phi'(x)=f(x)$ 或 $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Notes：

① $\frac{d}{dx}\int_a^{\varphi(x)} f(t)\,dt = f[\varphi(x)]\cdot\varphi'(x)$

② $\frac{d}{dx}\int_{\varphi_1(x)}^{\varphi_2(x)} f(t)\,dt = f[\varphi_2(x)]\cdot\varphi_2'(x) - f[\varphi_1(x)]\cdot\varphi_1'(x)$

例1. $\lim_{x\to0}\frac{\int_0^x\sin^2t\,dt-\frac{1}{3}x^3}{x^5} = \lim_{x\to0}\frac{\sin^2x-x^2}{5x^4} = \frac{1}{5}\lim_{x\to0}\frac{\sin x+x}{x}\times\frac{\sin x-x}{x^3}$

$= \frac{2}{5}\lim_{x\to0}\frac{\cos x-1}{3x^2} = -\frac{1}{15}$

表达式 是不允许有 x

例2. $f(x)$ 连续. $\varphi(x)=\int_0^x(x-t)f(t)\,dt$，求 $\varphi''(x)$

解：$\varphi(x) = x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt$

$\varphi'(x) = \int_0^x f(t)\,dt + xf(x) - xf(x) = \int_0^x f(t)\,dt$

$\varphi''(x) = f(x)$

例3. $f(x)$连续. $\varphi(x) = \int_0^x t f(x^2-t^2)\,dt$.

求 $\varphi'(x)$.

解: 1° $\varphi(x) = \int_0^x t f(x^2-t^2)\,dt$

$= -\frac{1}{2}\int_0^x f(x^2-t^2)\,d(x^2-t^2) \xlongequal{x^2-t^2=u} -\frac{1}{2}\int_{x^2}^0 f(u)\,du$

$= \frac{1}{2}\int_0^{x^2} f(u)\,du$

2° $\varphi'(x) = \frac{1}{2} \times f(x^2) \times 2x = x f(x^2)$

例4. $f(x)$连续, $f(0)=0$, $f'(0)=\pi$

$\lim\limits_{x\to 0} \frac{\int_0^x t f(x-t)\,dt}{x^3}$

解: $\int_0^x t f(x-t)\,dt \xlongequal{x-t=u} \int_x^0 (x-u) f(u)(-du) = \int_0^x (x-u) f(u)\,du$

$= x\int_0^x f(u)\,du - \int_0^x u f(u)\,du$

原式 $= \lim\limits_{x\to 0} \frac{x\int_0^x f(u)\,du - \int_0^x u f(u)\,du}{x^3} = \lim\limits_{x\to 0} \frac{\int_0^x f(u)\,du}{3x^2}$

$= \lim\limits_{x\to 0} \frac{f(x)}{6x} = \frac{1}{6}\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \frac{1}{6} f'(0) = \frac{\pi}{6}$

Th2. (N.-L.) $\int_a^b f(x)\,dx = F(b) - F(a)$

证: $F'(x) = f(x)$ $\Phi(x) = \int_a^x f(t)\,dt$.

$\Phi'(x) = f(x)$

$\Rightarrow$ $F(x)$, $\Phi(x)$ 为$f(x)$的原函数.

$\therefore F(x) - \Phi(x) = C_0$

$F(a) - \Phi(a) = C_0$ $F(b) - \Phi(b) = C_0$

$\Rightarrow F(a) - \Phi(a) = F(b) - \Phi(b)$

$\because \Phi(a) = 0$

$\therefore F(a) = F(b) - \Phi(b) \Rightarrow \Phi(b) = F(b) - F(a)$

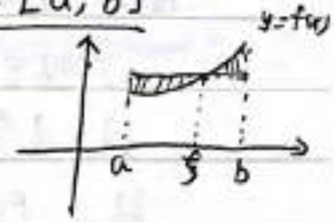
$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$.

四 定积分的一般性质:

1. $\int_a^b [k_1 f(x) + k_2 g(x)]\,dx = k_1\int_a^b f(x)\,dx + k_2\int_a^b g(x)\,dx$
2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
3. $\int_a^b 1\,dx = b - a$

4. ①（积分中值定理）设 $f(x)\in C[a,b]$，则 $\exists\ \xi\in[a,b]$

使 $\int_a^b f(x)dx = f(\xi)(b-a)$



证：$\because f(x)\in C[a,b]$

$\therefore \exists\ m, M$，

$m\le f(x)\le M$

$m(b-a)\le \int_a^b f(x)dx \le M(b-a)$

$\Rightarrow\quad m\le \frac{1}{b-a}\int_a^b f(x)dx \le M$

$\exists\ \xi\in[a,b]$，使 $f(\xi)=\frac{1}{b-a}\int_a^b f(x)dx$

$\Rightarrow \int_a^b f(x)dx = f(\xi)(b-a)$　　积分区间长度和2为倒数关系.

例1、$f(x)\in C[0,1]$，$(0,1)$ 内可导，$f(1)=2\int_0^{\frac{1}{2}}f(x)dx$.

证：$\exists\ \xi\in(0,1)$，$f'(\xi)=0$

证：1° $f(1)=2\times f(c)(\frac{1}{2}-0)$，$c\in[0,\frac{1}{2}]$

$\Rightarrow f(c)=f(1)$

2° $\exists\ \xi\in(c,1)\subset(0,1)$，使 $f'(\xi)=0$.

注：$f(x)\in C[a,b]$.

$f(x)$ 在 $[a,b]$ 上平均值为

$\bar{f}=\frac{\int_a^b f(x)dx}{b-a}$

②（积分中值定理推广）设 $f(x)\in C[a,b]$　则 $\exists\ \xi\in(a,b)$

使 $\int_a^b f(x)dx = f(\xi)(b-a)$.

若用积分中值定理，(c)不能取到0.

证：令 $F(x)=\int_a^x f(t)dt$.　$F'(x)=f(x)$　　$(a<\xi<b)$

$\int_a^b f(x)dx = F(b)-F(a) = F'(\xi)(b-a) = f(\xi)(b-a)$

例2. $f(x)\in C[0,2]$，$(0,2)$ 内可导. $2f(0)=\int_0^2 f(x)dx$

证 $\exists\ \xi\in(0,2)$，$f'(\xi)=0$

证 令 $F(x)=\int_0^x f(t)dt$　　$F'(x)=f(x)$

$\int_0^2 f(x)dx = F(2)-F(0) = F'(c)(2-0) = 2f(c)$　$(0<c<2)$

$f(0)=f(c)$

$\exists\ \xi\in(0,c)\subset(0,2)$　使 $f'(\xi)=0$

例3：$f(x)\in C[a,b]$，(a,b)内二阶可导

$f(a)=f(b)=\int_a^b f(x)dx=0$

证：$\exists\xi\in(a,b)$，$f''(\xi)=0$

记：$F(x)=\int_a^x f(t)dt$　$F'(x)=f(x)$

$F(a)=F(b)=0$，$\exists c\in(a,b)$，$F'(c)=0\Rightarrow f(c)=0$

$f(a)=f(c)=f(b)=0$

$\exists\xi_1\in(a,c)$，$\xi_2\in(c,b)$.

使 $f'(\xi_1)=0$，$f'(\xi_2)=0$

$\exists\xi\in(\xi_1,\xi_2)\subset(a,b)$ 使 $f''(\xi)=0$

5. ① $f(x)\geq 0\ (a\leq x\leq b)\Rightarrow\int_a^b f(x)dx\geq 0$

② $f(x)\geq g(x)\ (a\leq x\leq b)\Rightarrow\int_a^b f(x)dx\geq\int_a^b g(x)dx$

③ $\left|\int_a^b f(x)dx\right|\leq\int_a^b|f(x)|dx$　$(a<b)$

五、积分法.

(一) 换元积分法.

1. $\int_0^4 e^{\sqrt{x}}dx\xlongequal{\sqrt{x}=t}2\int_0^2 te^t dt=2\int_0^2 t\,d(e^t)=2(te^t|_0^2-\int_0^2 e^t dt)$

(二) 分部积分法.

$\int_a^b u\,dv=uv|_a^b-\int_a^b v\,du$.

六、特殊性质.

Notes:

① $\int_{-a}^0\xlongequal{x=-t}\int_0^a$

如：证 $\int_{-a}^0 f(x)dx=\int_0^a f(-x)dx$

证：$\int_{-a}^0 f(x)dx\xlongequal{x=-t}\int_a^0 f(-t)(-dt)=\int_0^a f(-t)dt=\int_0^a f(-x)dx$

② $\int_a^b\xlongequal{x+t=a+b}\int_a^b$

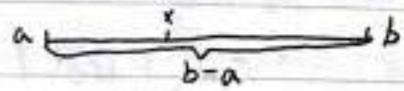
如：证：$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

证：$\int_a^b f(x)dx\xlongequal{x+t=a+b}\int_b^a f(a+b-t)(-dt)=\int_a^b f(a+b-t)dt$

$=\int_a^b f(a+b-x)dx$

③ $\int_a^{a+T} \xlongequal[\quad]{x-a=t} \int_0^T$

④ $\int_a^b \xlongequal[\quad]{x=a+(b-a)t} \int_0^1$



$x=a+(b-a)t$

例. 证: $\int_a^b f(x)dx = (b-a)\int_0^1 f[a+(b-a)x]\,dx$

证: $\int_a^b f(x)dx \xlongequal{x=a+(b-a)t} \int_0^1 f[a+(b-a)t](b-a)dt$

$= (b-a)\int_0^1 f[a+(b-a)x]\,dx$.

1.(对称性质) $f(x)\in C[-a,a]$, 则 $\int_{-a}^{a} f(x)dx = \int_0^a [f(x)+f(-x)]dx$

证: 左 $=\int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$

$\int_{-a}^{0} f(x)dx \xlongequal{x=-t} \int_a^0 f(-t)(-dt) = \int_0^a f(-t)dt = \int_0^a f(-x)dx$

2. ① $\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$

证: $\int_0^{\frac{\pi}{2}} f(\sin x)dx \xlongequal{x+t=\frac{\pi}{2}} \int_{\frac{\pi}{2}}^0 f(\cos t)(-dt) = \int_0^{\frac{\pi}{2}} f(\cos t)\,dt$

$= \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$

例1. 求 $I=\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx$

解. $I=\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}dx$

$2I = \frac{\pi}{2} \Rightarrow I=\frac{\pi}{4}$

特别地, $I_n=\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

记 $\begin{cases} I_n = \frac{n-1}{n} I_{n-2} \\ I_0 = \frac{\pi}{2} \\ I_1 = 1 \end{cases}$

例2 ① $\int_0^{\frac{\pi}{2}} \sin^{10} dx = I_{10} = \frac{9}{10} I_8 = \frac{9}{10}\times\frac{7}{8} I_6 = \frac{9}{10}\times\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$

② $\int_0^{\frac{\pi}{2}} \cos^{11} x = I_{11} = \frac{10}{11}\times\frac{8}{9}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times 1$

例3. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}dx$

解. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}dx = \int_0^{\frac{\pi}{4}} \left(\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right)dx$

$= \int_0^{\frac{\pi}{4}} \frac{2}{1-\sin^2 x}dx = 2\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$

$= 2\tan x \Big|_0^{\frac{\pi}{4}} = 2$

例4. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x}{1+e^x} dx = \int_0^{\frac{\pi}{2}} \left(\frac{\sin^4 x}{1+e^x} + \frac{\sin^4 x}{1+e^{-x}} \right) dx$

$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{1+e^x} + \frac{e^x}{e^x+1} \right) \sin^4 x \, dx$

$= \int_0^{\frac{\pi}{2}} \sin^4 x \, dx = \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

sint $t\in[-\frac{\pi}{2}, \frac{\pi}{2}]$ 找
cost $t\in[0,\pi]$ 找

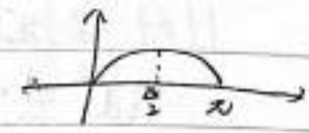
例5. $\int_0^1 x^2 \sqrt{1-x^2} \, dx \xlongequal{x=\sin t} \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t \, dt$

$= \int_0^{\frac{\pi}{2}} \sin^2 t (1-\sin^2 t) dt$

$= I_2 - I_4 = \frac{1}{2} \times \frac{\pi}{2} - \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{16}$

② $\int_0^{\pi} f(\sin x) dx = 2\int_0^{\frac{\pi}{2}} f(\sin x) dx$

$\int_0^{\pi} f(|\cos x|) dx = 2\int_0^{\frac{\pi}{2}} f(\cos x) dx$



3、f(x) 以T为周期

① $\int_a^{a+T} f(x) dx = \int_0^T f(x) dx$

证: $\int_a^{a+T} f(x) dx = \int_a^0 f(x) dx + \int_0^T f(x) dx + \int_T^{a+T} f(x) dx$

$\int_T^{a+T} f(x) dx \xlongequal{x-T=t} \int_0^a f(t+T) dt = \int_0^a f(t) dt = \int_0^a f(x) dx$

如: $\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^2 x \, dx = \int_0^{\pi} \sin^2 x \, dx = 2 \times I_2 = 2 \times \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{2}$

② $\int_0^{nT} f(x) dx = n\int_0^T f(x) dx$

七、反常积分（广义积分）

正常积分：$\begin{cases} \text{① 积分区间有限} \\ \text{② } f(x) \text{在} [a,b] \text{上连续或有限个第一类间断点.} \end{cases}$

(一) 区间无限.

1. $f(x) \in C[a,+\infty)$ 　 $\int_a^{+\infty} f(x) dx$

def- $\int_a^b f(x) dx = F(b) - F(a)$

$\begin{cases} \text{① } F(b)-F(a) \text{ 与 } \int_a^{+\infty} f(x) dx \text{ 不同} \\ \text{② } \lim\limits_{b\to+\infty} [F(b)-F(a)] \text{ 与 } \int_a^{+\infty} f(x) dx \text{ 同} \end{cases}$

$\lim\limits_{b\to+\infty} [F(b)-F(a)] \begin{cases} = A. \quad \int_a^{+\infty} f(x) dx = A \\ \text{不存在. 　发散.} \end{cases}$

如：$\int_1^{+\infty}\frac{1}{1+x^2}dx$

1° $\int_1^b \frac{1}{1+x^2}dx = \arctan x\Big|_1^b = \arctan b-\frac{\pi}{4}$

2° $\because \lim\limits_{b\to+\infty}(\arctan b-\frac{\pi}{4})=\frac{\pi}{4}$

$\therefore \int_1^{+\infty}\frac{1}{1+x^2}dx=\frac{\pi}{4}$

判别法一 $\lim\limits_{x\to+\infty}x^{\alpha}\cdot f(x)=C$. $(\neq 0)$

$\begin{cases}\text{收敛}, & \alpha>1\\ \text{发散}, & \alpha\le 1\end{cases}$

如：$\int_1^{+\infty}\frac{\sqrt{x}}{1+x^2}dx$?

$\because \lim\limits_{x\to+\infty}x^{\frac{3}{2}}\frac{\sqrt{x}}{1+x^2}=1$ 且 $\alpha=\frac{3}{2}>1$

$\therefore \int_1^{+\infty}\frac{\sqrt{x}}{1+x^2}dx$ 收敛

2. $f(x)\in C(-\infty,a]$. $\int_{-\infty}^{a}f(x)dx$

def — $\int_b^a f(x)dx = F(a)-F(b)$

$\lim\limits_{b\to-\infty}[F(a)-F(b)]\begin{cases}=A. & \int_{-\infty}^a f(x)dx=A\\ \text{不存在、发散.}\end{cases}$

3. $f(x)\in C(-\infty,+\infty)$. $\int_{-\infty}^{+\infty}f(x)dx$? 拆

$\int_{-\infty}^{+\infty}f(x)dx$ 收敛 $\Leftrightarrow$ $\int_{-\infty}^{a}f(x)dx$ 与 $\int_a^{+\infty}f(x)$ 皆收敛.

Q. $\int_{-\infty}^{+\infty}\frac{x}{1+x^2}dx \overset{?}{=} 0$ ×

对 $\int_0^{+\infty}\frac{x}{1+x^2}dx$

$\because \lim\limits_{x\to+\infty}x^1\frac{x}{1+x^2}=1$ 且 $\alpha=1\le 1$

$\therefore \int_0^{+\infty}\frac{x}{1+x^2}dx$ 发散

(二) 区间有限

1. $f(x)\in C(a,b]$ 且 $f(a+0)=\infty$ $\int_a^b f(x)dx$

def — $\forall \varepsilon>0$, $\int_{a+\varepsilon}^{b}f(x)dx = F(b)-F(a+\varepsilon)$

(① $F(b)-F(a+\varepsilon)$ 与 $\int_a^b f(x)dx$ 不同

② $\lim\limits_{\varepsilon\to0^+}[F(b)-F(a+\varepsilon)]$ 与 $\int_a^b f(x)dx$ 同)

$\lim\limits_{\varepsilon\to0^+}[F(b)-F(a+\varepsilon)]\begin{cases}=A & \int_a^b f(x)dx=A\\ \text{不存在, 发散}\end{cases}$

例1. $\int_1^2 \frac{dx}{x\sqrt{x-1}}$?

解: $\forall \varepsilon>0$, $\int_{1+\varepsilon}^2 \frac{dx}{x\sqrt{x-1}} = 2\int_{1+\varepsilon}^2 \frac{d(\sqrt{x-1})}{1+(\sqrt{x-1})^2} = 2\arctan\sqrt{x-1}\Big|_{1+\varepsilon}^2$

$= 2(\frac{\pi}{4} - \arctan\sqrt{\varepsilon})$

$\because \lim\limits_{\varepsilon\to0^+} 2(\frac{\pi}{4} - \arctan\sqrt{\varepsilon}) = \frac{\pi}{2}$

$\therefore \int_1^2 \frac{dx}{x\sqrt{x-1}} = \frac{\pi}{2}$

判别法:

$\lim\limits_{x\to a^+} (x-a)^{\alpha} f(x) = C_0$ $\begin{cases} 收敛, & \alpha<1 \\ 发散, & \alpha\geq1 \end{cases}$

如: $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}}$

解: $\because \lim\limits_{x\to0^+} (x-0)^{\frac{1}{2}} \frac{1}{\sqrt{x(1-x)}} = 1$ 且 $\alpha = \frac{1}{2} < 1$

$\therefore$ 收敛

$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} = 2\int_0^{\frac{1}{2}} \frac{d(\sqrt{x})}{\sqrt{1-(\sqrt{x})^2}} = 2\arcsin\sqrt{x}\Big|_0^{\frac{1}{2}} = \frac{\pi}{2}$

2、设 $f(x) \in C[a,b)$ 且 $f(b-0) = \infty$ $\int_a^b f(x)dx$

def - $\forall \varepsilon>0$, $\int_a^{b-\varepsilon} f(x)\,dx = F(b-\varepsilon) - F(a)$

$\lim\limits_{\varepsilon\to0^+} [F(b-\varepsilon)-F(a)]$ $\begin{cases} = A & \int_a^b f(x)dx = A \\ 不存在、发散 \end{cases}$

如: $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

1° $\forall \varepsilon>0$, $\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}dx = \arcsin x\Big|_0^{1-\varepsilon} = \arcsin(1-\varepsilon)$

2° $\because \lim\limits_{\varepsilon\to0^+} \arcsin(1-\varepsilon) = \frac{\pi}{2}$

$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$

判别法:

$\lim\limits_{x\to b^-} (b-x)^{\alpha} f(x) = C_0$ $\begin{cases} 收敛 & \alpha<1 \\ 发散 & \alpha\geq1 \end{cases}$

例2. 求 $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

解: 1° $\because \lim\limits_{x\to0^+} (x-0)^{\frac{1}{2}} \frac{1}{\sqrt{x(1-x)}} = 1$ 且 $\alpha = \frac{1}{2} < 1$

又 $\because \lim\limits_{x\to1^-} (1-x)^{\frac{1}{2}} \frac{1}{\sqrt{x(1-x)}} = 1$ 且 $\alpha = \frac{1}{2} < 1$

$\therefore$ 收敛. 2° $\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = 2\int_0^1 \frac{d(\sqrt{x})}{\sqrt{1-(\sqrt{x})^2}} = 2\arcsin\sqrt{x}\Big|_0^1 = \pi$

3. $f(x)\in C[a,c)\cup(c,b]$ 且 $\lim_{x\to c}f(x)=\infty$

$\int_a^b f(x)dx$ 收敛 $\Leftrightarrow$ $\int_a^c f(x)dx$ 与 $\int_c^b f(x)dx$ 皆收敛

逆：Γ函数.

1. 定义 — $\int_0^{+\infty} x^{\alpha-1}e^{-x}dx \triangleq \Gamma(\alpha)$

如：$\int_0^{+\infty} x^4 e^{-x}dx = \Gamma(5)$

战如：$\int_0^{+\infty}\sqrt{x}e^{-x}dx = \Gamma(\frac{3}{2})$

2. 性质：

① $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$；

② $\Gamma(n+1)=n!$

③ $\Gamma(\frac{1}{2})=\sqrt{\pi}$

例1. $\int_0^{+\infty}x^3e^{-x}dx=\Gamma(4)=3!=6$

例2. $\int_0^{+\infty}x^{\frac{3}{2}}e^{-x}dx=\Gamma(\frac{5}{2})=\Gamma(\frac{3}{2}+1)=\frac{3}{2}\Gamma(\frac{3}{2})$

$=\frac{3}{2}\Gamma(\frac{1}{2}+1)=\frac{3}{2}\times\frac{1}{2}\Gamma(\frac{1}{2})=\frac{3}{4}\sqrt{\pi}$

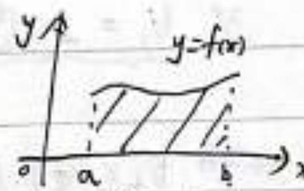
例3. $\int_0^{+\infty}x^5e^{-x^2}dx=\frac{1}{2}\int_0^{+\infty}x^4e^{-x^2}d(x^2)$

$\xlongequal{x^2=t}\frac{1}{2}\int_0^{+\infty}t^2e^{-t}d(t)=\frac{1}{2}\Gamma(3)=1$

定积分应用书写 $\begin{cases}\text{几何应用 (一、二、三)}\\ \text{物理应用 (二、三)}\end{cases}$

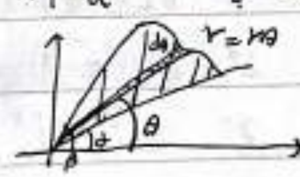
几何应用：

一、面积：



1. $L: y=f(x)\geq 0\ (a\leq x\leq b)$

$A=\int_a^b f(x)dx$



2. $L: r=r(\theta)\ (\alpha\leq\theta\leq\beta)$

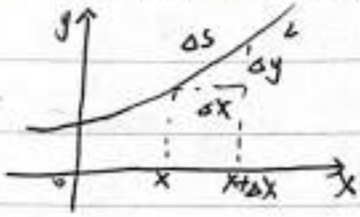
1° 取 $[\theta,\theta+d\theta]\subset[\alpha,\beta]$；

2° $dA=\frac{1}{2}r^2(\theta)d\theta$；

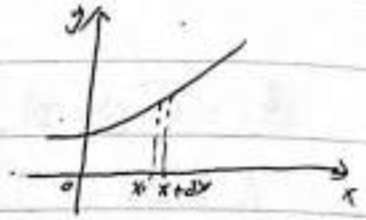
3° $A=\frac{1}{2}\int_\alpha^\beta r^2(\theta)d\theta$.

3.

$$A=\frac{1}{2}\int_{\alpha}^{\beta}[r_2^2(\theta)-r_1^2(\theta)]\,d\theta$$

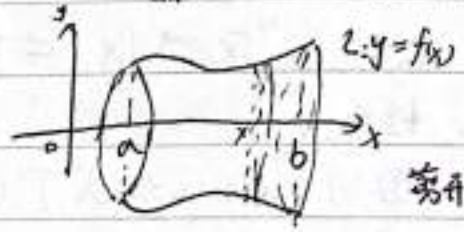


— $\Delta s \approx \sqrt{(\Delta x)^2+(\Delta y)^2}$



$\frac{dy}{dx}$ dy　　$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{1+f'^2(x)}\,dx$

4.



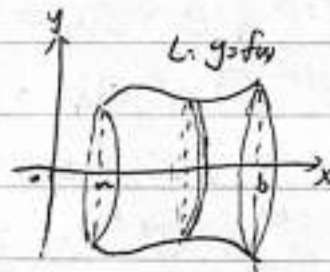
1° 取 $[x, x+dx]\subset[a,b]$;

2° $dA = 2\pi|f(x)|\cdot ds$.

$\quad = 2\pi|f(x)|\cdot\sqrt{1+f'^2(x)}\,dx$;

3° $A = 2\pi\int_a^b|f(x)|\cdot\sqrt{1+f'^2(x)}\,dx$.

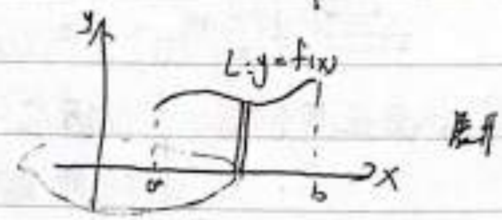
二、体积:



1、1° 取 $[x, x+dx]\subset[a,b]$:

2°. $dv = \pi f^2(x)\,dx$;

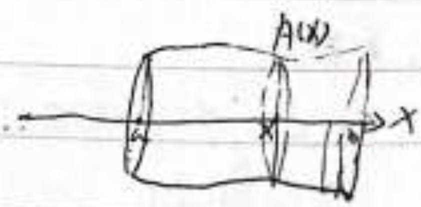
3°. $V_x = \pi\int_a^b f^2(x)\,dx$.



2、1° 取 $[x, x+dx]\subset[a,b]$;

2° $dv = 2\pi|x|\cdot|f(x)|\cdot dx$;

3° $V_y = 2\pi\int_a^b|x|\cdot|f(x)|\cdot dx$



3、1° $\forall[x, x+dx]\subset[a,b]$

2° $dv = A(x)\,dx$

3° $V = \int_a^b A(x)\,dx$

三、弧长.

1. $L: y=f(x)\ (a\le x\le b)$.

1° 取 $[x, x+dx]\subset[a,b]$;

$2^\circ\quad ds=\sqrt{1+f'^2(x)}\,dx$;　　　　$ds=\sqrt{(dx)^2+(dy)^2}$

$3^\circ\quad l=\int_a^b\sqrt{1+f'^2(x)}\,dx$.

2. $L: \begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}\ (\alpha\le t\le\beta)$.

1° 取 $[t, t+dt]\subset[\alpha,\beta]$.

$2^\circ\ ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt=\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$.

$3^\circ\ l=\int_\alpha^\beta\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$.

注:

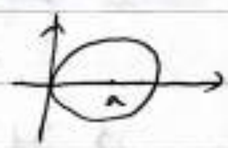
1. 圆: $(a>0)$

① $x^2+y^2=a^2 \Leftrightarrow r=a$

② $x^2+y^2=2ax \Leftrightarrow (x-a)^2+y^2=a^2$

$\Updownarrow$

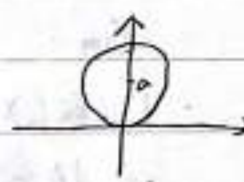
$r=2a\cos\theta$



③ $x^2+y^2=2ay \Leftrightarrow x^2+(y-a)^2=a^2$

$\Updownarrow$

$r=2a\sin\theta$

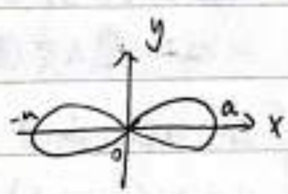


2. 双纽线

$(x^2+y^2)^2=a^2(x^2-y^2)$

$\Updownarrow$

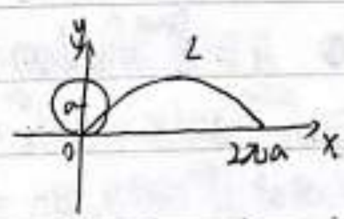
$r^2=a^2\cos 2\theta$



$0\le\theta\le\frac{\pi}{4}$

3. 摆线.

$L: \begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}\ (0\le t\le 2\pi)$



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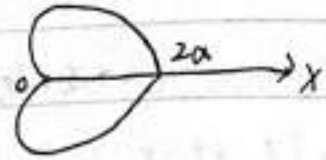
④

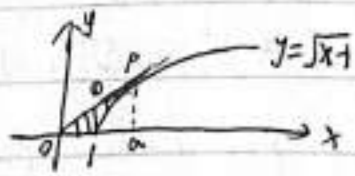
4. 心形线

L: $r = a(1+\cos\theta)$.



例题：

1. L: $y=\sqrt{x-1}$



解，① 令 $P(a, \sqrt{a-1})$

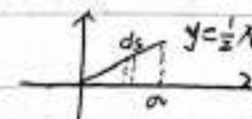
由 $\frac{1}{2\sqrt{a-1}} = \frac{\sqrt{a-1}}{a} \Rightarrow a=2 \Rightarrow P(2,1)$.

∴ 切线 $y=\frac{1}{2}x$.

② $A = 1-\int_1^2 \sqrt{x-1}\,dx = 1-\int_1^2 (x-1)^{\frac{1}{2}}d(x-1)$

$= 1-\frac{2}{3}(x-1)^{\frac{3}{2}}\Big|_1^2 = 1-\frac{2}{3}=\frac{1}{3}$

③ $S_{外}$:



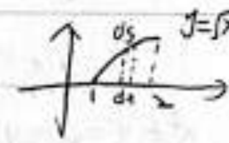
1° 取 $[x, x+dx] \subset [0,2]$;

2° $dA = 2\pi y \cdot ds = 2\pi \cdot \frac{1}{2}x \cdot \sqrt{1+\frac{1}{4}}\,dx$

$= \frac{\sqrt{5}}{2}\pi x\,dx$

3° $S_{外} = \frac{\sqrt{5}}{2}\pi\int_0^2 x\,dx = \sqrt{5}\pi$.

$S_{内}$:



1° 取 $[x, x+dx] \subset [1,2]$;

2° $dA = 2\pi y \cdot ds = 2\pi\cdot\sqrt{x-1}\times\sqrt{1+\frac{1}{4(x-1)}}\,dx$

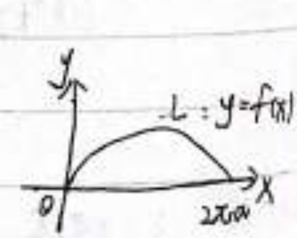
$= \pi\sqrt{4x-3}\,dx$.

3° $S_{外} = \pi\int_1^2\sqrt{4x-3}\,dx = \frac{\pi}{4}\int_1^2(4x-3)^{\frac{1}{2}}d(4x-3)$

$= \frac{\pi}{4}\times\frac{2}{3}(4x-3)^{\frac{3}{2}}\Big|_1^2 = \frac{\pi}{6}(5\sqrt{5}-1)$

2. L: $\begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases}$ $(a>0,\ 0\le t\le 2\pi)$

① 求面积



解：① $A = \int_0^{2\pi a} f(x)\,d(x)$

$= \int_0^{2\pi a} y\,dx = \int_0^{2\pi} a(1-\cos t)\,a(1-\cos t)\,dt = a^2\int_0^{2\pi}(2\sin^2\frac{t}{2})^2 dt$

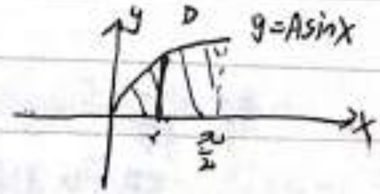
$= 8a^2\int_0^{2\pi}\sin^4\frac{t}{2}\,2d\frac{t}{2} = 8a^2\int_0^{\pi}\sin^4 t\,dt = 8a^2\times 2\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = 3\pi a^2$

② $V_x = \pi\int_0^{2\pi a} f^2(x)\,dx = \pi\int_0^{2\pi a} y^2\,dx = \pi\int_0^{2\pi} a^2(1-\cos t)^2\, a(1-\cos t)\,dt$

$= \pi a^3\int_0^{2\pi}(2\sin^2\frac{t}{2})^3\,dt = 16\pi a^3\int_0^{2\pi}\sin^6\frac{t}{2}\,d\frac{t}{2} = 16\pi a^3\int_0^{\pi}\sin^6 t\,dt$

$= 32\pi a^3\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$

3. $V_x = V_y$ $\quad A=?$



解: ① $V_x = \pi\int_0^{\frac{\pi}{2}} y^2\,dx$

$= \pi A^2\int_0^{\frac{\pi}{2}}\sin^2 x\,dx$

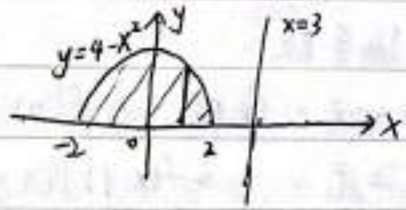
$= \pi A^2\times\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi^2}{4}A^2$

② 取 $[x, x+dx]\subset[0,\frac{\pi}{2}]$

$dV_y = 2\pi x\cdot y\cdot dx = 2\pi x\cdot A\sin x\,dx$

$V_y = 2\pi A\int_0^{\frac{\pi}{2}} x\sin x\,dx =$

4. 解:



绕 $x=3$ 求体积

1° 取 $[x, x+dx]\subset[-2, 2]$

2° $dV = 2\pi(3-x)\cdot y\cdot dx$

$= 2\pi(3-x)(4-x^2)\,dx$

3° $V = 2\pi\int_{-2}^{2}(3-x)(4-x^2)\,dx$

第六章 多元函数微分学

一、定义

1. 极限

一元 — If $\forall\varepsilon>0$, $\exists\delta>0$, 当 $0<|x-a|<\delta$

$|f(x)-A|<\varepsilon$

$\lim\limits_{x\to a} f(x) = A$

★ $\lim\limits_{x\to a} f(x)\ \exists \iff f(a-0),\ f(a+0)\ \exists$ 且相等

二元 — $z=f(x,y)$ $\quad (x,y)\in D$ $\quad M_0(x_0,y_0)$

If $\forall\varepsilon>0$, $\exists\delta>0$, 当 $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$ 时

$|f(x,y)-A|<\varepsilon$ 称 $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = A$.

例1. $\lim\limits_{\substack{x\to 0\\ y\to 2}}(1+xy^2)^{\frac{1}{\sin x}}$ (1^{∞})

解: 原 $=\lim\limits_{\substack{x\to 0\\ y\to 2}}\left[(1+xy^2)^{\frac{1}{xy^2}}\right]^{\frac{xy^2}{\sin x}}=e^{\lim\limits_{\substack{x\to 0\\ y\to 2}}\frac{x}{\sin x}y^2}=e^4$

例2. $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ $\lim\limits_{\substack{x\to 0\\ y\to 0}}f(x,y)$?

解: $\because \lim\limits_{\substack{x\to 0\\ y=x}}f(x,y)=\frac{1}{2}\neq\lim\limits_{\substack{x\to 0\\ y=-x}}f(x,y)=-\frac{1}{2}$

$\therefore \lim\limits_{\substack{x\to 0\\ y\to 0}}f(x,y)$ 不存在

2. 连续

一元 - If $\lim\limits_{x\to a}f(x)=f(a)$ 称 $f(x)$ 在 $x=a$ 连续

$f(x)$ 在 $x=a$ 连续 $\Leftrightarrow f(a-0)=f(a+0)=f(a)$

二元 - $z=f(x,y)$ $((x,y)\in D)$. $M_0(x_0,y_0)\in D$.

If $\lim\limits_{\substack{x\to x_0\\ y\to y_0}}f(x,y)=f(x_0,y_0)$ 称 $f(x,y)$ 在 (x_0,y_0) 处连续

3. 偏导数

一元: 导数 - $f'(a)=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}$

二元: $z=f(x,y)$ $((x,y)\in D)$, $M_0(x_0,y_0)\in D$

$\Delta z_x=f(x_0+\Delta x,y_0)-f(x_0,y_0)$ - $f(x,y)$ 在 M_0 关于 x 偏增量

$\Delta z_y=f(x_0,y_0+\Delta y)-f(x_0,y_0)$ - $f(x,y)$ 在 M_0 关于 y 偏增量

$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$ - $f(x,y)$ 在 M_0 处的全增量

If $\lim\limits_{\Delta x\to 0}\frac{\Delta z_x}{\Delta x}$ $\exists$, 称 $f(x,y)$ 在 M_0 关于 x 可偏导.

极限值为 $f(x,y)$ 在 M_0 关于 x 的偏导数. 记 $\frac{\partial z}{\partial x}\Big|_{M_0}$. $f'_x(x_0,y_0)$

Note: $f'_x(x_0,y_0)\triangleq\lim\limits_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}\triangleq\lim\limits_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$

$f'_y(x_0,y_0)\triangleq\lim\limits_{\Delta y\to 0}\frac{\Delta z_y}{\Delta y}=\lim\limits_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$

4. 多元函数(全)微分

一元 - $y=f(x)$ $(x\in D)$, $x_0\in D$

$\Delta y=f(x_0+\Delta x)-f(x_0)\overset{\text{If}}{=\!=}A\Delta x+o(\Delta x)$.

称 $f(x)$ 在 $x=x_0$ 可微. $A\Delta x\triangleq dy|_{x=x_0}=A\,dx$

$\begin{cases}\text{可导}\Leftrightarrow\text{可微}\\ A=f'(x_0)\\ df(x)=f'(x)\,dx\end{cases}$

二元一 $z=f(x,y)\ ((x,y)\in D)$. $M_0(x_0,y_0)\in D$.

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$

$$\overset{if}{=\!=} A\Delta x + B\Delta y + o(\rho),\ 其中\ \rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$$

称 $f(x,y)$ 在 M_0 处可(全)微

$$A\Delta x + B\Delta y \triangleq dz|_{M_0}\ 或\ dz|_{M_0} = A\,dx + B\,dy$$

二、理论.

1、几大特性关系：

可偏导：$f'_x(x,y)$，$f'_y(x,y)\ \exists$

连续可偏导：$f'_x(x,y)$，$f'_y(x,y)$ 连续

连续 $\overset{①}{\nLeftrightarrow}$ 可偏导

连续 ← 可微 → 可偏导

可微 ⇑② 连续可偏导

↓ 可微

证明部分：

①"可微 ⇒ 连续"

设 $f(x,y)$ 在 M_0 可微，则 $\Delta z = f(x,y) - f(x_0,y_0)$ ~~$=A\Delta x$~~

$$= A(x-x_0) + B(y-y_0) + o\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$$

$\because \lim\limits_{\substack{x\to x_0\\ y\to y_0}} \Delta z = 0$.

$\therefore \lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f(x_0,y_0)$ 即 $f(x,y)$ 在 M_0 连续

②"可微 ⇒ 可偏导"

设 $f(x,y)$ 在 M_0 可微则

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = A\Delta x + B\Delta y + o(\rho)$$

取 $\Delta y=0$ $\quad \Delta z_x = A\Delta x + o(\Delta x) \Rightarrow \frac{\Delta z_x}{\Delta x} = A + \frac{o(\Delta x)}{\Delta x}$

$\because \lim\limits_{\Delta x\to 0}\frac{\Delta z_x}{\Delta x} = A$ $\quad\therefore z=f(x,y)$ 在 M_0 处对 x 可偏导 $f'_x(x_0,y_0)=A$

同理 $z=f(x,y)$ 在 M_0 处关于 y 可偏导，且 $f'_y(x_0,y_0)=B$.

反例部分：

①"连续 $\nRightarrow$ 可偏导"

例1. $z=f(x,y)=\sqrt{x^2+y^2}$ 在 $(0,0)$ 连续，讨论 $f(x,y)$ 在 $(0,0)$ 可偏导性

解. $\lim\limits_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0}\frac{|x|}{x}$ 不存在. $f(x,y)$ 在 $(0,0)$ 对 x 不可偏导

同理 $f(x,y)$ 在 $(0,0)$ 对 y 不可偏导.

②"可偏导 ⇏ 连续"

例2. $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

研究 $f(x,y)$ 在 $(0,0)$ 处连续性，可偏导性.

解：$\lim\limits_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to 0}\frac{\frac{0}{x^2}-0}{x}=\lim\limits_{x\to 0}\frac{0}{x^3}=0$

$\Rightarrow f'_x(0,0)=0$

同理 $f_y'(0,0)=0$

$\because \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ 不存在 而 $f(0,0)=0$

$\therefore f(x,y)$ 在 $(0,0)$ 不连续.

2 若 $z=f(x,y)$ 二阶连续可偏导，则

$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$

如：$z=x^3-x^2y+2xy^2-y^3$

$\frac{\partial z}{\partial x}=3x^2-2xy+2y^2 \qquad \frac{\partial z}{\partial y}=-x^2+4xy-3y^2$

$\frac{\partial^2 z}{\partial x^2}=6x-2y \qquad \frac{\partial^2 z}{\partial y^2}=4x-6y$

$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=-2x+4y \qquad \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=-2x+4y$

其.

三、求导类型：

(一) 显函数求偏导

例1. $z=\arctan^2\frac{x+y}{1-xy}$，求 $\frac{\partial z}{\partial x}$

解：$\frac{\partial z}{\partial x}=2\arctan\frac{x+y}{1-xy}\cdot\frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\frac{(1-xy)+(x+y)\cdot y}{(1-xy)^2}$

例2. $z=(1+x^2y)^{x^2+y^2}$ 求 $\frac{\partial z}{\partial x}$

解：$z=e^{(x^2+y^2)\ln(1+x^2y)}$

$\frac{\partial z}{\partial x}=\underbrace{e^{(x^2+y^2)\ln(1+x^2y)}}_{=(1+x^2y)^{x^2+y^2}}\cdot\left[2x\cdot\ln(1+x^2y)+(x^2+y^2)\cdot\frac{2xy}{1+x^2y}\right]$

(二) 复合函数求偏导.

① $z=f(x^2+y^2)$: $z=f(u)$, $u=x^2+y^2$.

② $z=f(x^2+y^2, xy)$: $z=f(u,v)$ $\begin{cases} u=x^2+y^2 \\ v=xy \end{cases}$

$\frac{\partial f}{\partial u} \triangleq f_1'=f_1'(u,v)$. $(f_u', f_u'(u,v))$

$\frac{\partial f}{\partial v} \triangleq f_2'=f_2'(u,v)$ $(f_v', f_v'(u,v))$

$\frac{\partial^2 f}{\partial u^2} \triangleq f_{11}''$, $\frac{\partial^2 f}{\partial u \partial v}=f_{12}''$

③ $z=f(xy, x+y, x^2+y^2)$; $z=f(u,v,w)$. $\begin{cases} u=xy \\ v=x+y \\ w=x^2+y^2 \end{cases}$

例1. $z=f(x^2+y^2)$. f=阶可导. 求 $\frac{\partial^2 z}{\partial x \partial y}$

解: $\frac{\partial z}{\partial x}=2x\, f'(x^2+y^2)$

$\frac{\partial^2 z}{\partial x \partial y}=2x\cdot 2y\, f''(x^2+y^2)$

例2. $z=f(x^2+y^2, xy)$. f=阶连续可偏导, 求 $\frac{\partial^2 z}{\partial x \partial y}$.

解: $\frac{\partial z}{\partial x}=2xf_1'+yf_2'$

$\frac{\partial^2 z}{\partial x \partial y}=2x\cdot(2yf_{11}''+xf_{12}'')+f_2'+y(2yf_{21}''+xf_{22}'')$

$=4xyf_{11}''+2(x^2+y^2)f_{12}''+f_2'+xyf_{22}''$

例3. ~~$z=f(x,y$~~ $z=f(xy, x^2+y^2, x^2)$ f=阶连续可偏导

求 $\frac{\partial^2 z}{\partial x \partial y}$

解: $\frac{\partial z}{\partial x}=yf_1'+2xf_2'+2xf_3'$

$\frac{\partial^2 z}{\partial x \partial y}=f_1'+y(xf_{11}''+2yf_{12}'')+2x(xf_{21}''+2yf_{22}'')$

$+2x(xf_{31}''+2yf_{32}'')$

(三) 隐函数(组)求偏导.

方程 — 约束条件.

$\begin{cases} \text{参约束的变量} - \text{函数} \\ \text{不参约束的变量} - \text{自变量} \end{cases}$

① $F(x,y)=0 \Rightarrow$ 一个一元 $y=f(x)$

$\Downarrow$

$y=\varphi(x)$

② $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \Rightarrow$ 二个一元. $\Rightarrow \begin{cases} y=y(x) \\ z=z(x) \end{cases}$

③ $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \Rightarrow$ 二个二元.

$\Downarrow$

$\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$

例1. $x^2+2xz^2+y^2z=5$ 求 $\frac{\partial z}{\partial x}$?

解: 1° $x^2+2xz^2+y^2z=5 \Rightarrow z=z(x,y)$.

2° 两边对x求偏导

$2x+2z^2+2x\cdot 2z\cdot\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial x}=0$

$\Rightarrow \frac{\partial z}{\partial x}=$

例2. $\begin{cases} x-2y+3z=1 \\ x^2+y^2+xz+z^2=5. \end{cases}$ 求 $\frac{dz}{dx}$

1° $\begin{cases} x-2y+3z=1 \\ x^2+y^2+xz+z^2=5 \end{cases} \Rightarrow \begin{cases} y=y(x) \\ z=z(x) \end{cases}$

2° $\begin{cases} 1-2\frac{dy}{dx}+3\frac{dz}{dx}=0 \\ 2x+2y\cdot\frac{dy}{dx}+z+x\frac{dz}{dx}+2z\frac{dz}{dx}=0 \end{cases}$

$\Rightarrow \begin{cases} 2\frac{dy}{dx}-3\frac{dz}{dz}=1 \\ 2y\frac{dy}{dx}+(x+2z)\frac{dz}{dx}=-2x-2y-z \end{cases}$

$D=\begin{vmatrix} 2 & -3 \\ 2y & x+2z \end{vmatrix}=2x+6y+4z$

$D_1=\begin{vmatrix} 1 & -3 \\ -2x-2y-z & x+2z \end{vmatrix}$. $D2=$

$\frac{dy}{dx}=\frac{D_1}{D}$ $\frac{dz}{dx}=\frac{D_2}{D}$

例3. $\begin{cases} xu+yv=x+y \\ x^2+y^2+u^2+v^2=10 \end{cases}$ 求 $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$.

解: 1° $\begin{cases} xu+yv=x+y \\ x^2+y^2+u^2+v^2=10 \end{cases} \Rightarrow \begin{cases} u=u(x,y) \\ v=v(x,y). \end{cases}$

2° $\begin{cases} u+x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial x}=1 \\ 2x+2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0 \end{cases}$

$\Rightarrow \begin{cases} x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial x}=1-u \\ u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=-x \end{cases}$

应用: 代数应用: 无条件极值与条件极值.
几何应用: (数学一).

一元: $y=f(x)$

1° $x\in D$;

2° $f'(x)$ $\begin{cases} =0 \ (\text{驻点}) \\ \text{不存在}. \end{cases}$;

3° 判别法:

法一: ① $\begin{cases} x<x_0, f'(x)<0 \\ x>x_0, f'(x)>0 \end{cases} \Rightarrow x_0$ 极小点

② $\begin{cases} x<x_0, f'>0 \\ x>x_0, f'<0 \end{cases} \Rightarrow x_0$ 为极大点.

法二: $f'(x_0)=0$, $f''(x_0)$ $\begin{cases} >0, \text{小} \\ <0, \text{大} \end{cases}$

一、无条件极值.

$z=f(x,y)$. $(x,y)\in D$, D为开区域.

1° $\begin{cases} \frac{\partial z}{\partial x}=\cdots=0 \\ \frac{\partial z}{\partial y}=\cdots=0 \end{cases} \Rightarrow \begin{cases} x=? \\ y=? \end{cases}$ (驻点)

2° 设 $(x,y)=(x_0,y_0)$ 为一个驻点

$A=f''_{xx}(x_0,y_0)$. $B=f''_{xy}(x_0,y_0)$ $C=f''_{yy}(x_0,y_0)$

$AC-B^2\begin{cases}>0, & (x_0,y_0) \text{ 为极值点}. \begin{cases}A>0, \text{极小点}\\A<0, \text{极大点}\end{cases}\\<0, & (x_0,y_0)\text{不是极值点}.\end{cases}$

例1. $z=x^3-3x^2-9x+y^2-4y+3$. 求极值点与极值。

解：1° $\begin{cases}z'_x=3x^2-6x-9=0\\z'_y=2y-4=0\end{cases}\Rightarrow\begin{cases}x=-1\\y=2\end{cases}$ 或 $\begin{cases}x=3\\y=2\end{cases}$

2° $\frac{\partial^2 z}{\partial x^2}=6x-6$ $\frac{\partial^2 z}{\partial x\partial y}=0$ $\frac{\partial^2 z}{\partial y^2}=2$

当 $(x,y)=(-1,2)$ 时. $A=-12$ $B=0$ $C=2$

$AC-B^2<0$ $\therefore(-1,2)$不是极值点

当 $(x,y)=(3,2)$ 时, $A=12$ $B=0$ $C=2$.

$AC-B^2>0$ 且$A>0$ $\therefore(3,2)$ 为极小点.

极小值 $z=f(3,2)$

二. 条件极值. (仅讲二元).

$z=f(x,y)$ s.t. $\varphi(x,y)=0$ (약束)

法一：$\varphi(x,y)=0\Rightarrow y=h(x)$. 代入 $z=f(x,y)$.

$z=f[x,h(x)]$

即条件极值化为一元函数极值.

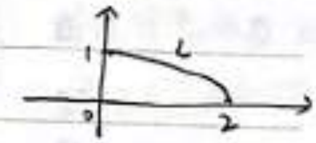
法二：Lagrange 乘数法：

1°. $F(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y)$.

2°. $\begin{cases}F'_x=f'_x+\lambda\varphi'_x=0 & ①\\F'_y=f'_y+\lambda\varphi'_y=0 & ②\\F'_\lambda=\varphi(x,y)=0 & ③\end{cases}$

$\Rightarrow\begin{cases}x=?\\y=?\end{cases}$

例1. 求 $z=\frac{2}{xy}$ 在 L: $\frac{x^2}{4}+y^2=1$ $(x\geq 0, y\geq 0)$ 上的最小值.



解: 1°. $F=\frac{2}{xy}+\lambda(\frac{x^2}{4}+y^2-1)$.

2°. $$\begin{cases} F'_x=-\frac{2}{x^2y}+\frac{\lambda}{2}x=0 & ① \\ F'_y=-\frac{2}{xy^2}+2\lambda y=0 & ② \\ F'_\lambda=\frac{x^2}{4}+y^2-1=0 & ③ \end{cases}$$ (消λ)

由①;②⇒ $y=\frac{x}{2}$

代入③ $$\begin{cases} x=\sqrt{2} \\ y=\frac{\sqrt{2}}{2} \end{cases}$$

∴ 当 $\begin{cases} x=\sqrt{2} \\ y=\frac{\sqrt{2}}{2} \end{cases}$ 时. $z=\frac{2}{xy}$ 最小, 最小值为 $z_{min}=\frac{2}{\sqrt{2}\cdot\frac{\sqrt{2}}{2}}=2$.

例2. $u=u(x,y)$ 满足: $\begin{cases} du=2x\,dx-2y\,dy \\ u(0,0)=3. \end{cases}$

①求 $u(x,y)$; ②求 $u(x,y)$ 在 D: $\frac{x^2}{4}+y^2\leq 1$ 上的 m, M.

解: ①法一: 由 $du=2x\,dx-2y\,dy$ 得

$\frac{\partial u}{\partial x}=2x$. $\frac{\partial u}{\partial y}=-2y$.

$\frac{\partial u}{\partial x}=2x \Rightarrow u=x^2+\varphi(y)$

$\therefore \frac{\partial u}{\partial y}=\varphi'(y)=-2y \Rightarrow \varphi(y)=-y^2+C$

$\therefore u=x^2-y^2+C$

$\because u(0,0)=3 \quad \therefore C=3.$

$\therefore u(x,y)=x^2-y^2+3$

法二: $du=2x\,dx-2y\,dy=d(x^2)-d(y^2)=d(x^2-y^2)$

$\Rightarrow u=x^2-y^2+C.$

$\because u(0,0)=3 \quad \therefore C=3$

$\therefore u=x^2-y^2+3.$

② 当 $\frac{x^2}{4}+y^2<1$ 时.

由 $\begin{cases}\frac{\partial u}{\partial x}=2x=0\\ \frac{\partial u}{\partial y}=-2y=0\end{cases} \Rightarrow \begin{cases}x=0\\ y=0.\end{cases}$ $u(0,0)=3$;

当 $\frac{x^2}{4}+y^2=1$

令 $F=(x^2-y^2+3)+\lambda(\frac{x^2}{4}+y^2-1)$

$\begin{cases}F_x'=2x+\frac{\lambda}{2}x=0 & ①\\ F_y'=-2y+2\lambda y=0 & ②\\ F_\lambda'=\frac{x^2}{4}+y^2-1=0 & ③\end{cases}$ $y(\lambda-1)=0$

当 $y=0$ 时, 代入③, $\begin{cases}x=-2\\ y=0\end{cases}$, $\begin{cases}x=2\\ y=0\end{cases}$

当 $\lambda=1$ 时. 代入 ① $\Rightarrow x=0$ 代入③

$\begin{cases}x=0\\ y=-1\end{cases}$ $\begin{cases}x=0\\ y=1.\end{cases}$

$u(\pm 2,0)=7$ $u(0,\pm 1)=2$

$\therefore$ $m=2$, $M=7$.

第八章 微分方程

Part I 一阶微分方程. 一种数解法

Notes:

① 含导数或微分方程 称 微分方程

$2x+y=4$ ×

$\frac{dy}{dx}+2xy=0 \Rightarrow y=y(x)$

一阶微分方程.

$yy''-y'^2=0 \Rightarrow$ $y'=\frac{dy}{dx}$. $y''=\frac{d^2y}{dx^2}$.

$y=y(x)$

二阶微分方程

② 满足微分方程的函数称为微分方程的解.

如: $y''-3y'+2y=0$

$y_1=e^x$ 为特解.

$y_2=e^{2x}$ 为特解.

$y_3=C_1e^x+C_2e^{2x}$ 为方程通解.

特解 — 不含任意常数的解
通解 — 解的表达式中所含的相互独立的任意常数的个数与微分方程阶数相同

一、可分离变量的D.E 微分方程

def — 设 $\frac{dy}{dx}=f(x,y)$. $(*)$

If $f(x,y)=\varphi_1(x)\varphi_2(y)$. 称 $(*)$ 为可分离 —

解法: $\frac{dy}{dx}=\varphi_1(x)\varphi_2(y) \Rightarrow \frac{dy}{\varphi_2(y)}=\varphi_1(x)dx$

$\Rightarrow \int\frac{dy}{\varphi_2(y)}=\int\varphi_1(x)dx+C$

例1. 求 $\frac{dy}{dx}=1+x+y^2+xy^2$ 通解.

解: $\frac{dy}{dx}=(1+x)(1+y^2)$

$\int\frac{dy}{1+y^2}=\int(1+x)dx+C \Rightarrow \arctan y=x+\frac{1}{2}x^2+C$

$\therefore y=\tan(x+\frac{x^2}{2}+C)$.

例2. 求 $y'+2xy=0$ 通解.

解: $\frac{dy}{dx}=-2xy$

① $y=0$ 为方程的解

② $y\neq0$ 时.

$\frac{dy}{y}=-2x\,dx$

$\int \Rightarrow \ln|y|=-x^2+C_0$ $\quad -\infty<C_0<+\infty \quad e^{C_0}\in(0,+\infty)$

$\Rightarrow |y|=e^{C_0}\cdot e^{-x^2} \Rightarrow y=\pm e^{C_0}\cdot e^{-x^2}$

令 $\pm e^{C_0}=C$ 则 $y=Ce^{-x^2}$ $(C\neq0)$

$\therefore$ 通解 $y=Ce^{-x^2}$ (C为任意常数).

二. 齐次微分方程

def- 设 $\frac{dy}{dx}=f(x,y)$. (*)

If $f(x,y)=\varphi(\frac{y}{x})$. 称(*)为齐次微分方程.

解法: $\frac{dy}{dx}=\varphi(\frac{y}{x})$.

令 $\frac{y}{x}=u \Rightarrow y=xu. \Rightarrow \frac{dy}{dx}=u+x\frac{du}{dx}$ 代入

$u+x\frac{du}{dx}=\varphi(u) \Rightarrow x\frac{du}{dx}=\varphi(u)-u$

$\int\frac{du}{\varphi(u)-u}=\int\frac{dx}{x}+C \Rightarrow u=\varphi(x,c)$

$\therefore\ y=x\varphi(x,c)$.

例1. 求 $\frac{dy}{dx}=2\frac{y}{x}-1$ 通解.

解: 令 $\frac{y}{x}=u$, 则 $u+x\frac{du}{dx}=2u-1$

$\Rightarrow \frac{du}{u-1}=\frac{dx}{x}$

$\Rightarrow \ln(u-1)=\ln x+\ln c$

$\Rightarrow u-1=cx \Rightarrow u=cx+1$

$\therefore$ 通解: $y=cx^2+x$

例2. 求 $xdy-(y+\sqrt{x^2+y^2})dx=0$ $(x>0)$ 通解.

解. $\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x} \Rightarrow \frac{dy}{dx}=\frac{y}{x}+\sqrt{1+(\frac{y}{x})^2}$

令 $u=\frac{y}{x}$. 则 $u+x\frac{du}{dx}=u+\sqrt{1+u^2}$

$\frac{du}{\sqrt{1+u^2}}=\frac{dx}{x}$.

$\Rightarrow \ln(u+\sqrt{u^2+1})=\ln x+\ln c$

$\Rightarrow u+\sqrt{u^2+1}=cx$

$\because -u+\sqrt{u^2+1}=\frac{1}{cx}$

$\therefore u=\frac{1}{2}(cx-\frac{1}{cx})$

$\therefore$ 通解 $y=\frac{1}{2}(cx^2-\frac{1}{c})$

三、一阶齐次线性微分方程

def 一 形如 $\frac{dy}{dx}+p(x)y=0$ (*)

称为一阶齐次线性微分方程

解法： $\frac{dy}{dx}=-p(x)y$

①$y=0$，为方程的解；

②$y\neq0$时，

$$\frac{dy}{y}=-p(x)dx$$

$$\ln|y|=-\int_0^x p(x)dx+C_0$$

$$\Rightarrow |y|=e^{C_0}e^{-\int_0^x p(x)dx} \Rightarrow y=\pm e^{C_0}e^{-\int_0^x p(x)dx}$$

令 $\pm e^{C_0}=C$ $(C\neq0)$

$\therefore y=Ce^{-\int_0^x p(x)dx}$ $(C\neq0)$

$\therefore$ 通解 $y=Ce^{-\int_0^x p(t)dt}$

$\frac{dy}{dx}+P(x)y=0$ 通解为 $y=Ce^{-\int_0^x P(t)dt}$

例1. 求 $\frac{dy}{dx}+2xy=0$ 通解

解：通解 $y=Ce^{-\int_0^x 2xdx}=Ce^{-x^2}$ （C为任意常数）

例2. 求 $\frac{dy}{dx}+y\tan x=0$ 通解.

解：通解：$y=Ce^{-\int\tan x dx}=Ce^{\ln\cos x}$

$=C\cos x$ （C为任意常数）

四、一阶非齐次线性微分方程

def 一 形如 $\frac{dy}{dx}+P(x)y=Q(x)$ 易经

称为一阶非齐次线性微分方程 变化 规律

解法： $\frac{dy}{dx}+P(x)y=0 \overset{(*)}{\Rightarrow} y=Ce^{-\int P(x)dx}$

$\frac{dy}{dx}+P(x)y=Q(x)$ (**)

常数变易法： 即 $C\to C(x)$

令(**)通解为 $y=\underline{C(x)}e^{-\int P(x)dx}$.

代入 $C'(x)e^{-\int P(x)dx}\cancel{-P(x)C(x)e^{-\int P(x)dx}}+\cancel{P(x)C(x)e^{-\int P(x)dx}}=Q(x)$

$C'(x)e^{-\int P(x)dx}=Q(x)$, $C'(x)=Q(x)e^{\int P(x)dx}$

$\therefore C(x)=\int Q(x)e^{\int P(x)dx}dx+C$

记. ∴ $\frac{dy}{dx} + P(x)y = Q(x)$ 通解为.

$y = [\int Q(x) e^{\int P(x)dx} dx + C] e^{-\int P(x)dx}$

例1. 求 $\frac{dy}{dx} - \frac{2}{x}y = -1$ 通解.

解: 法一: $\frac{dy}{dx} - \frac{2}{x}y = 0$ 通解为

$y = ce^{-\int -\frac{2}{x}dx} = cx^2$

令原方程通解为 $y = C(x)x^2$ 代入

$C'(x)x^2 + C(x)\cdot 2x - \frac{2}{x}C(x)x^2 = -1$

$C'(x) = -\frac{1}{x^2}$ ∴ $C(x) = \frac{1}{x} + C$

∴通解 $y = cx^2 + x$

法二:

$y = [\int Q(x) e^{\int P(x)dx} dx + C] e^{-\int P(x)dx}$

$= [\int (-1) e^{\int -\frac{2}{x}dx} dx + C] e^{-\int -\frac{2}{x}dx}$

$= (\frac{1}{x} + C) x^2$

$= cx^2 + x$

例2. $f(x)$连续. $f(x) - 2\int_0^x f(x-t)dt = e^{2x}$

求$f(x)$

解: 1° $\int_0^x f(x-t)dt \xlongequal{x-t=u} \int_x^0 f(u)(-du) = \int_0^x f(u)du$

2° $f(x) - 2\int_0^x f(u)du = e^{2x}$

$\Rightarrow f'(x) - 2f(x) = 2e^{2x}$

3° $f(x) = [\int 2e^{2x} e^{\int -2dx} dx + C] e^{-\int -2dx}$

$= (2x + C) e^{2x}$

∴ $f(x) = ce^{2x} + 2xe^{2x}$

4° ∵ $f(0) = 1$ ∴ $C = 1$

∴ $f(x) = (2x+1)e^{2x}$

Part Ⅱ 可降阶的高阶微分方程（一、二）

一、$y^{(n)}=f(x)$

如：$y''=3x^2$ $y'=x^3+C_1$ $y=\frac{1}{4}x^4+C_1x+C_2$

二、$f(x,y',y'')=0$ （缺 y）

令 $y'=p$ $y''=\frac{dp}{dx}$ 代入

$f(x,p,\frac{dp}{dx})=0 \Rightarrow p=\varphi(x,C_1)$ 即 $y'=\varphi(x,C_1)$

$\therefore y=\int\varphi(x,C_1)dx+C_2$

例1. 求 $xy''+2y'=0$ 通解.

解：法一：令 $y'=p$ $y''=\frac{dp}{dx}$ 代入

$x\frac{dp}{dx}+2p=0 \Rightarrow \frac{dp}{dx}+\frac{2}{x}p=0$

$p=C_1e^{-\int\frac{2}{x}dx}=C_1e^{-\ln x^2}=C_1e^{\ln\frac{1}{x^2}}=\frac{C_1}{x^2}$

即 $y'=\frac{C_1}{x^2}$

$\therefore y=-\frac{C_1}{x}+C_2$

法二：$xy''+2y'=0 \Rightarrow x^2y''+2xy'=0$

$\Rightarrow (x^2y')'=0$

$\therefore x^2y'=C_1$

$y'=\frac{C_1}{x^2}$

$\therefore y=-\frac{C_1}{x}+C_2$

三、$f(y,y',y'')=0$ （缺 x）

令 $y'=p$ $y''=\frac{dp}{dx} \Rightarrow f(y,p,\frac{dp}{dx})=0$?

$y''=\frac{dp}{dx}=\frac{dy}{dx}\frac{dp}{dy}=p\frac{dp}{dy}$ 代入.

$f(y,p,p\frac{dp}{dy})=0$

例2. 求 $yy''-y'^2=0$ 满足 $y(0)=1$，$y'(0)=1$ 的特解.

解：令 $y'=p$. $y''=p\frac{dp}{dy}$，代入

$yp\frac{dp}{dy}-p^2=0$

$\because p\neq0 \quad \therefore y\frac{dp}{dy}-p=0$

$\Rightarrow \frac{dp}{dy}-\frac{1}{y}p=0$

$p=C_1e^{-\int\frac{1}{y}dy}=C_1y$

即 $y'=C_1y$ $\because y(0)=1$，$y'(0)=1$ $\therefore C_1=1$ $\therefore y'=y$ 即 $\frac{dy}{dx}-y=0$

$\because$ $y = C_1 e^{-\int -dx} = C_2 e^x$

$\because$ $y(0)=1$ $\therefore C_2 = 1$

$\therefore$ $y = e^x$

Part Ⅲ. 高阶线性 D·E

一、基本理论

(一) def.

$$y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_{n-1}(x) y' + a_n(x) y = 0 \quad (*)$$

n阶齐次线性微分方程

$$y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_{n-1}(x) y' + a_n(x) y = f(x) \quad (**)$$

n阶非齐线性微分方程

I $f(x) = f_1(x) + f_2(x)$，则

$$y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_{n-1}(x) y' + a_n(x) y = f_1(x) \quad (**)'$$

$$y^{n} + a_1(x) y^{(n-1)} + \cdots + a_{n-1}(x) y' + a_n(x) y = f_2(x) \quad (**)''$$

(二) 结构

1. $\varphi_1(x), \cdots, \varphi_s(x)$ 为 $(*)$ 的解。
   则 $y = C_1\varphi_1(x) + \cdots + C_s\varphi_s(x)$ 为 $(*)$ 的解
2. $\varphi_1(x), \cdots, \varphi_s(x)$ 为 $(**)$ 的解。
   ① $C_1\varphi_1(x) + \cdots + C_s\varphi_s(x)$ 为 $(*)$ 的解。$\Leftrightarrow C_1 + \cdots + C_s = 0$
   ② $C_1\varphi_1(x) + \cdots + C_s\varphi_s(x)$ 为 $(**)$ 解。$\Leftrightarrow C_1 + \cdots + C_s = 1$.
3. $\varphi_1(x), \varphi_2(x)$ 为 $(**)$ 解，则 $\varphi_2(x) - \varphi_1(x)$ 为 $(*)$ 解。
4. $\varphi_1(x), \varphi_2(x)$ 为 $(*), (**)$ 解，则 $\varphi_1(x) + \varphi_2(x)$ 为 $(**)$ 解。
5. $\varphi_1(x), \varphi_2(x)$ 为 $(**)', (**)''$ 的解，
   则 $\varphi_1(x) + \varphi_2(x)$ 为 $(**)$ 解

二、特殊情形

(一) $y''+py'+qy=0$ (p, q are 常数)

二阶常系数齐次线性DE.

1° $\lambda^2+p\lambda+p=0$

2° ① $\Delta>0 \Rightarrow \lambda_1\neq\lambda_2$ 实

通解 $y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$；

例1. 由 $y''-y'-6y=0$ 通解.

解：$\lambda^2-\lambda-6=0 \Rightarrow \lambda_1=-2\neq\lambda_2=3$

通解：$y=C_1e^{-2x}+C_2e^{3x}$

② $\Delta=0 \Rightarrow \lambda_1=\lambda_2$ 实

通解：$y=(C_1+C_2x)e^{\lambda_1 x}$

例2. 求 $y''-6y'+9y=0$ 通解

解. $\lambda^2-6\lambda+9=0 \Rightarrow \lambda_1=\lambda_2=3$

∴通解 $y=(C_1+C_2x)e^{3x}$

③ $\Delta<0 \Rightarrow \lambda_{1,2}=\alpha\pm i\beta$

通解，$y=e^{\alpha x}\cdot(C_1\cos\beta x+C_2\sin\beta x)$.

例3. 求 $y''+4y=0$ 通解.

$\lambda^2+4=0 \quad \lambda_{1,2}=\pm 2i$

通解 $y=C_1\cos 2x+C_2\sin 2x$.

例4. 求 $y''-2y'+2y=0$ 通解.

解：$\lambda^2-2\lambda+2=0 \Rightarrow \lambda_{1,2}=1\pm i$.

通解：$y=e^x(C_1\cos x+C_2\sin x)$.

(二) $y'''+py''+qy'+ry=0$ (p, q, r 为常数).

三阶常系数齐次线性D.E.

1° 特征方程. $\lambda^3+p\lambda^2+q\lambda+r=0$

2° ① $\lambda_1, \lambda_2, \lambda_3$ 实、单

通解 $y=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}+C_3e^{\lambda_3x}$

② $\lambda_1=\lambda_2\neq\lambda_3$ 实

通解: $y=(C_1+C_2x)e^{\lambda_1x}+C_3e^{\lambda_3x}$;

③ $\lambda_1=\lambda_2=\lambda_3$ 实

通解: $y=(C_1+C_2x+C_3x^2)e^{\lambda_1x}$

例1. 求 $y'''-y''-2y'=0$ 通解.

解: $\lambda^3-\lambda^2-2\lambda=0 \Rightarrow \lambda(\lambda+1)(\lambda-2)=0$

$\Rightarrow \lambda_1=0 \quad \lambda_2=-1 \quad \lambda_3=2$

通解: $y=C_1+C_2e^{-x}+C_3e^{2x}$

例2 求 $y'''-2y''+y'=0$ 通解

解: $\lambda^3-2\lambda^2+\lambda=0 \Rightarrow \lambda(\lambda-1)^2=0$

$\Rightarrow \lambda_1=0 \quad \lambda_2=\lambda_3=1$

通解 $y=C_1+(C_2+C_3x)e^x$.

例3. 求 $y'''-3y''+3y'-y=0$ 通解.

解. $\lambda^3-3\lambda^2+3\lambda-1=0 \Rightarrow (\lambda-1)^3=0$

$\Rightarrow \lambda_1=\lambda_2=\lambda_3=1$

∴ 通解 $y=(C_1+C_2x+C_3x^2)e^x$

④ $\lambda_1\in R$, $\lambda_{2,3}=\alpha\pm i\beta$

通解 $y=C_1e^{\lambda_1x}+e^{\alpha x}(C_2\cos\beta x+C_3\sin\beta x)$

(三) $y''+py'+qy=f(x)$ (p, q 是常数)

二阶常系数非齐次线性DE

1° $y''+py'+qy=0 \Rightarrow$ 通解

2° $y''+py'+qy=f(x) \Rightarrow$ 特解 + $y_0(x)$?

通 + 特 = $y''+py'+qy=f(x)$ 通解

型一：$f(x) = P_n(x)e^{kx}$

对应右边假设 $k \neq \lambda_1, \lambda_2$

例1. $y'' - 3y' + 2y = xe^{-x}$ 通解：

解：1° $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_1 = 1 \quad \lambda_2 = 2$

$y'' - 3y' + 2y = 0$ 通解 $y = C_1e^x + C_2e^{2x}$

2° 令 $y_0(x) = (ax+b)e^{-x}$ 代入． $a = \quad , \quad b = \quad .$

$y_0' = ae^{-x} - (ax+b)e^{-x} = (-ax+a-b)e^{-x}$

$y_0'' = -ae^{-x} + (ax-a+b)e^{-x} = (ax-2a+b)e^{-x}$

$ax - 2a + b - 3(-ax+a-b) + 2(ax+b) = x$

$6ax - 5a + 6b = x$

$$\begin{cases} 6a = 1 \\ -5a + 6b = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{6} \\ b = \frac{5}{36} \end{cases}$$

∴ 通解 $y = C_1e^x + C_2e^{2x} + (\frac{x}{6} + \frac{5}{36})e^{-x}$

例2 求 $y'' - 3y' + 2y = (3x-2)e^x$ 通解

对应右边假设 一次多项式 $k = 1 = \lambda_1 \neq \lambda_2$

解 1° $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 2$

$y'' - 3y' + 2y = 0 \Rightarrow y = C_1e^x + C_2e^{2x}$,

2° 令 $y_0(x) = x(ax+b)e^x = (ax^2 + bx)e^x$

代入 $a = -\frac{3}{2}, \quad b = -1$

$y_0'(x) = (2ax+b)e^x + (ax^2+bx)e^x = (ax^2 + 2ax + bx + b)e^x$

$y_0''(x) = (2ax + 2a + b)e^x + (ax^2 + 2ax + bx + b)e^x$

$= (ax^2 + 4ax + bx + 2a + 2b)e^x$

$(\cancel{ax^2} + 4ax + \cancel{bx} + 2a + 2b) - 3(\cancel{ax^2} + 2ax + \cancel{bx} + b) + 2(\cancel{ax^2} + \cancel{bx}) = 3x - 2$

$-2ax + 2a - b = 3x - 2$

$$\begin{cases} -2a = 3 \\ 2a - b = -2 \end{cases} \Rightarrow a = -\frac{3}{2} \quad b = -1$$

∴ 通解 $y = C_1e^x + C_2e^{2x} + (-\frac{3}{2}x^2 - x)e^x$

公众号 3·盖吉……

例3. 求 $y''-4y'+4y=(6x-1)e^{2x}$ 通解

$k=\lambda_1=\lambda_2$

解: 1° $\lambda^2-4\lambda+4=0 \Rightarrow \lambda_1=\lambda_2=2$

$y''-4y'+4y=0$ 通解 $y=(C_1+C_2x)e^{2x}$;

2° 令 $y_0(x)=x^2(ax+b)e^{2x}=(ax^3+bx^2)e^{2x}$

代入, $a=1$, $b=-\frac{1}{2}$

$\therefore$ 通解 $y=(C_1+C_2x)e^{2x}+(x^3-\frac{1}{2}x^2)e^{2x}$

例4. $f(x)$ 连续. $f(x)-4\int_0^x tf(x-t)\,dt=e^{2x}$ (*)

求 $f(x)$.

解: 1° $\int_0^x tf(x-t)\,dt \xlongequal{x-t=u} \int_x^0 (x-u)f(u)(-du)$

$=\int_0^x (x-u)f(u)\,du = x\int_0^x f(u)du-\int_0^x uf(u)\,du$

2° $f'(x)-4\int_0^x f(u)du=2e^{2x}$ (**)

3° $f''(x)-4f(x)=4e^{2x}$ (***)

$\lambda^2-4=0 \Rightarrow \lambda_1=-2,\ \lambda_2=2$

$f''(x)-4f(x)=0$ 通解 $f(x)=C_1e^{-2x}+C_2e^{2x}$

令 $f_0(x)=\cancel{x(ax+b)}\ axe^{2x}$ 代入 (***)

$f_0'(x)=ae^{2x}+2axe^{2x}=(2ax+a)e^{2x}$

$f_0''(x)=2ae^{2x}+(4ax+2a)e^{2x}=(4ax+4a)e^{2x}$

$4ax+4a-4ax=4$

$a=1$

$\therefore$ 通解 $y=C_1e^{-2x}+C_2e^{2x}+xe^{2x}$

4° $\because f(0)=1,\quad f'(0)=2.$

$\therefore \begin{cases} C_1+C_2=1 \\ \ldots\ldots \end{cases}$

第七章 重积分 $\begin{cases}\text{二重积分}\\ \text{三重积分 (一)}\end{cases}$

一、产生背景：

例子. D－xoy 面上有限闭区域，其面密度 $\rho(x,y)$，求 m.

$1°$. $D \Rightarrow \Delta\sigma_1, \Delta\sigma_1, \cdots, \Delta\sigma_n$;

$2°$. $\forall(\xi_i, \eta_i) \in \Delta\sigma_i$.

$\Delta m_i \approx \rho(\xi_i,\eta_i)\Delta\sigma_i$

$m \approx \sum_{i=1}^{n} \rho(\xi_i,\eta_i)\Delta\sigma_i$;

$3°$ 令 λ 为 $\Delta\sigma_1, \Delta\sigma_2, \cdots \Delta\sigma_n$ 直径最大者.

$m = \lim_{\lambda\to 0} \sum_{i=1}^{n} \rho(\xi_i,\eta_i)\Delta\sigma_i$.

二、def — D－xoy 面平面有界闭区域，$f(x,y)$ 在D上有界.

$1°$ $D \Rightarrow \Delta\sigma_1, \Delta\sigma_2, \cdots \Delta\sigma_n$;

$2°$ $\forall(\xi_i, \eta_i) \in \Delta\sigma_i$.

作 $\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i$;

$3°$ 令 λ 为 $\Delta\sigma_1, \Delta\sigma_2, \cdots \Delta\sigma_n$ 直径最大者.

若 $\lim_{\lambda\to 0} \sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i$ $\exists$

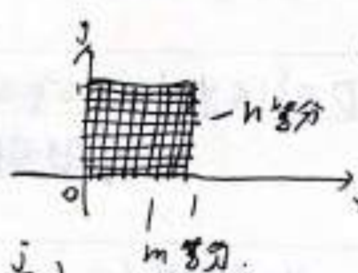
称 $f(x,y)$ 在D上可积，极限值称为 $f(x,y)$ 在D上的二重积分. 记 $\iint_D f(x,y)d\sigma$.

即 $\iint_D f(x,y)d\sigma \triangleq \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i$.

Notes:

设 $D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$

$f(x,y)$ 在D上可积.



$1°$

$2°$ 对 $[\frac{i-1}{m}, \frac{i}{m}]\times[\frac{j-1}{n}, \frac{j}{n}]$. 取 $(\frac{i}{m}, \frac{j}{n})$

$\Delta\sigma_{ij} = \frac{1}{mn}$

作 $\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f(\frac{i}{m}, \frac{j}{n})$;

$3°$ ∵ $\lambda\to 0 \Leftrightarrow m\to\infty, n\to\infty$

记 ① $\lim_{\substack{m\to\infty\\ n\to\infty}} \frac{1}{mn}\sum_{i=1}^{m}\cdot\sum_{j=1}^{n} f(\frac{i}{m}, \frac{j}{n}) = \iint_D f(x)d\sigma$.

② $\lim\limits_{n\to\infty}\frac{1}{n^2}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} f(\frac{i}{n},\frac{j}{n}) = \iint\limits_D f(x,y)\,d\sigma$.

例1 求 $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\frac{n}{(n+j)(n^2+j^2)}$

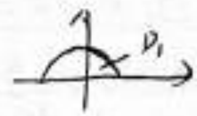
原式 $=\lim\limits_{n\to\infty}\frac{1}{n^2}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\frac{1}{(1+\frac{i}{n})(1+(\frac{j}{n})^2)}$

$=\iint\limits_D\frac{1}{(1+x)(1+y^2)}\,d\sigma$.

$(D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 1\})$.

三、性质:

1. $\iint\limits_D 1\,d\sigma = A$;



2、① D关于y轴对称，右 D_1，则

$\begin{cases} 若 f(-x,y)=-f(x,y) \Rightarrow \iint\limits_D f(x,y)\,d\sigma=0; \\ 若 f(-x,y)=f(x,y) \Rightarrow \iint\limits_D f(x,y)\,d\sigma=2\iint\limits_{D_1} f(x,y)\,d\sigma; \end{cases}$

② D关于X轴对称，上 D_1 则.

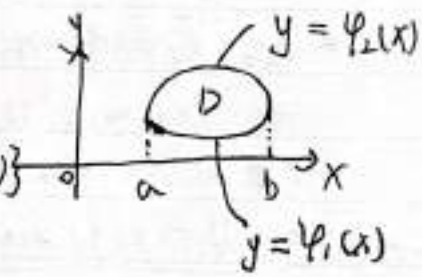
$\begin{cases} f(x,-y)=-f(x,y) \Rightarrow \iint\limits_D f(x,y)\,d\sigma=0. \\ f(x,-y)=f(x,y) \Rightarrow \iint\limits_D f(x,y)\,d\sigma=2\iint\limits_{D_1} f(x,y)\,d\sigma. \end{cases}$

③ D关于 y=X 对称，则.

$\iint\limits_D f(x,y)\,d\sigma=\iint\limits_D f(y,x)\,d\sigma$.

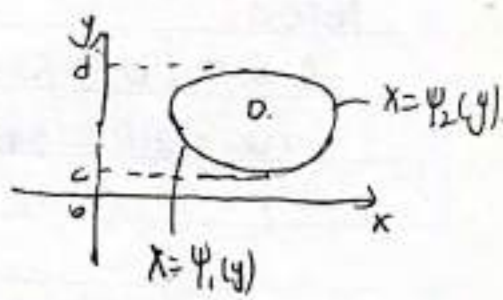
四、二重积分计算方法.

(一) 直角坐标法.



$D=\{(x,y)\mid a\le x\le b,\ \varphi_1(x)\le y\le\varphi_2(x)\}$

—X-型区域



$D=\{(x,y)\mid \psi_1(y)\le x\le\psi_2(y),\ c\le y\le d\}$

—Y-型区域

X-型区域下:

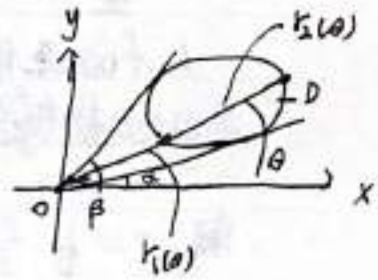
$\iint\limits_D f(x,y)\,d\sigma=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$.

Y-型下:

$\iint\limits_D f(x,y)\,d\sigma=\int_c^d dy\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$

(二) 极坐标法:

1° 特征: $\begin{cases} ① \text{区域 } D \text{ 边界含 } x^2+y^2. \\ ② f(x,y) \text{ 含 } x^2+y^2. \end{cases}$



2° 变换: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\alpha \le \theta \le \beta \qquad r_1(\theta) \le r \le r_2(\theta)$

3° $d\sigma = r\,dr\,d\theta$

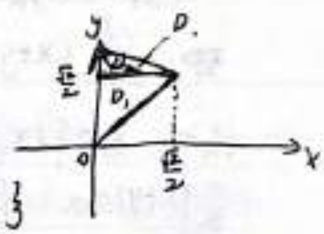
$\iint\limits_D f(x,y)\,d\sigma = \int_\alpha^\beta d\theta \int_{r_1(\theta)}^{r_2(\theta)} r\cdot f(r\cos\theta, r\sin\theta)\,dr$

型一: 改变积分次序

- 改变积分次序
- 次序不对无法计算
- 积分法不对

例1 改变次序:

$\int_0^{\frac{\sqrt{2}}{2}} dx \int_x^{\sqrt{1-x^2}} f(x,y)\,dy$.



解: 1° 画 D

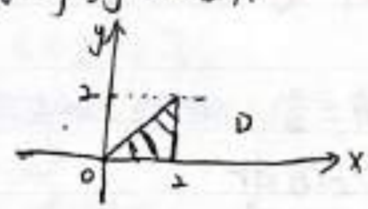
2° $D_1 = \{(x,y) \mid 0 \le x \le y,\ 0 \le y \le \frac{\sqrt{2}}{2}\}$

$D_2 = \{(x,y) \mid 0 \le x \le \sqrt{1-y^2}\ \ \frac{\sqrt{2}}{2} \le y \le 1\}$

3° 原式 $= \int_0^{\frac{\sqrt{2}}{2}} dy \int_0^y f(x,y)\,dx + \int_{\frac{\sqrt{2}}{2}}^1 dy \int_0^{\sqrt{1-y^2}} f(x,y)\,dx$.

例2. $\int_0^2 dy \int_y^2 e^{x^2} dx$

解:



Notes: 以下情况改次序

① $x^{2n} e^{\pm x^2} dx$

② $e^{\frac{1}{x}} dx$

③ $\begin{cases} \sin\frac{1}{x}\,dx \\ \cos\frac{1}{x}\,dx \end{cases}$

原式 $= \int_0^2 dx \int_0^x e^{x^2} dy$.

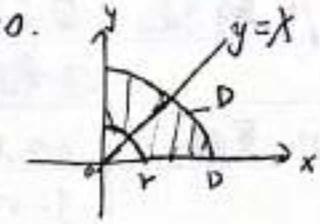
$= \int_0^2 e^{x^2} dx \int_0^x 1\,dy = \int_0^2 x e^{x^2} dx$

$= \frac{1}{2} e^{x^2}\Big|_0^2$

$= \frac{1}{2}(e^4 - 1)$

型二 计算

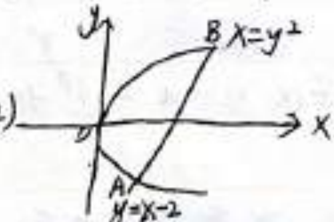
1. $f(x)$ 连续，$f(x)>0$，$a>0$，$b>0$。
$I=\iint \frac{af(x)+bf(y)}{f(x)+f(y)}\,d\sigma$?



解：$I=\iint_D \frac{af(y)+bf(x)}{f(y)+f(x)}\,d\sigma$

$2I=\iint(a+b)\,d\sigma=(a+b)\iint d\sigma=(a+b)\cdot\frac{\pi R^2-\pi r^2}{4}$

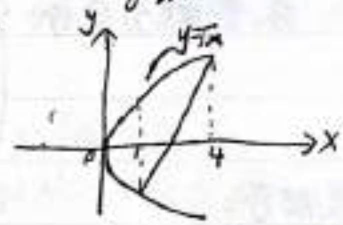
2. $\iint(x+y)\,d\sigma$



解：由 $\begin{cases}x=y^2\\ y=x-2\end{cases}\Rightarrow A(1,-1)$，$B(4,2)$

法一：$D_1=\{(x,y)\mid 0\le x\le 1, -\sqrt{x}\le y\le\sqrt{x}\}$

$D_2=\{(x,y)\mid 1\le x\le 4, x-2\le y\le\sqrt{x}\}$

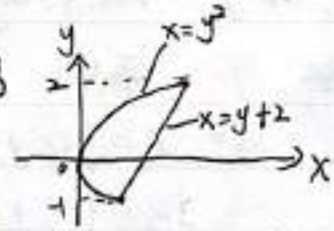


$\iint(x+y)\,d\sigma=\int_0^1dx\int_{-\sqrt{x}}^{\sqrt{x}}(x+y)\,dy$
$+\int_1^4dx\int_{x-2}^{\sqrt{x}}(x+y)\,dy$

如：$\int_{-\sqrt{x}}^{\sqrt{x}}(x+y)\,dy=x\int_{-\sqrt{x}}^{\sqrt{x}}1\,dy=2x\sqrt{x}$

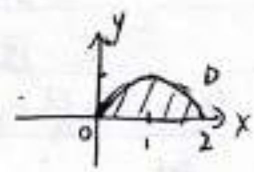
法二：$D=\{(x,y)\mid y^2\le x\le y+2, -1\le y\le 2\}$

$\iint(x+y)\,d\sigma=\int_{-1}^2dy\int_{y^2}^{y+2}(x+y)\,dx$



例3. $\iint x^2\,d\sigma$ $\quad D=x^2+y^2\le 2x\quad(y\ge 0)$

$D=(x-1)^2+y^2\le 1\quad(y\ge 0)$



解：令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ $\quad(0\le\theta\le\frac{\pi}{2},\ 0\le r\le 2\cos\theta)$.

$\iint_D x^2\,d\sigma=\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^3\cos^2\theta\,dr$

$=\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta\int_0^{2\cos\theta}r^3\,dr=4\int_0^{\frac{\pi}{2}}\cos^6\theta\,d\theta$

$=4\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}=\frac{5\pi}{8}$

例4. $\iint\frac{d\sigma}{\sqrt{x^2+y^2}}$

作图.
先奇偶,对称 解：令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ $\quad(0\le\theta\le\frac{\pi}{2},\ \frac{1}{\cos\theta+\sin\theta}\le r\le 1)$

$x+y=1\Rightarrow r\cos\theta+r\sin\theta=1$

$\iint_D\frac{d\sigma}{\sqrt{x^2+y^2}}=\int_0^{\frac{\pi}{2}}d\theta\int_{\frac{1}{\cos\theta+\sin\theta}}^1 1\,dr=\int_0^{\frac{\pi}{2}}\left(1-\frac{1}{\sin\theta+\cos\theta}\right)d\theta$.

$=\frac{\pi}{2}-\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\csc(\theta+\frac{\pi}{4})\,d(\theta+\frac{\pi}{4})=\frac{\pi}{2}-\frac{1}{\sqrt{2}}\times\ln\left|\csc(\theta+\frac{\pi}{4})-\cot(\theta+\frac{\pi}{4})\right|\Big|_0^{\frac{\pi}{2}}$

重积分续

一、三重积分.

(一) def — Ω — 有界闭几何体. $f(x,y,z)$ 在 Ω 上有界

$1°$ $\Omega \Rightarrow \Delta V_1, \Delta V_2, \cdots, \Delta V_n$;

$2°$ $\forall(\xi_i,\eta_i,\zeta_i)\in\Delta V_i$. 作

$$\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta V_i;$$

$3°$. λ 为 $\Delta V_1,\cdots,\Delta V_n$ 直径最大者.

若 $\lim\limits_{\lambda\to0}\sum\limits_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta V_i$ $\exists$

称此极限为 $f(x,y,z)$ 在 Ω 上的三重积分

记 $\iiint\limits_{\Omega} f(x,y,z)\,dV$

即 $\iiint\limits_{\Omega} f(x,y,z)\,dv \triangleq \lim\limits_{\lambda\to0}\sum\limits_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta V_i$.

二、性质.

1. $\iiint\limits_{\Omega} 1\,dv = V$

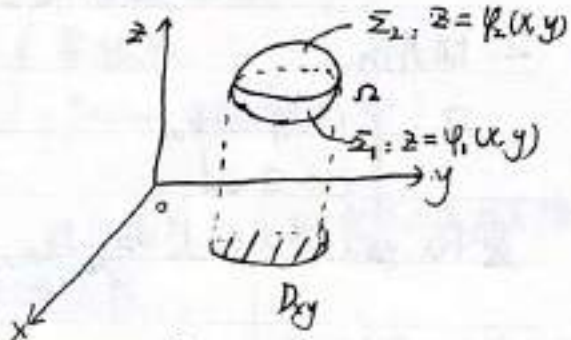
2. ① Ω 关于 xoy 面对称. 上 Ω_1

If $f(x,y,-z)=-f(x,y,z)$, 则 $\iiint\limits_{\Omega} f(x,y,z)dv=0$;

If $f(x,y,-z)=f(x,y,z)$, 则 $\iiint\limits_{\Omega} f\,dv = 2\iiint\limits_{\Omega_1} f\,dv$.

三、计算方法

(一) 直角坐标法.

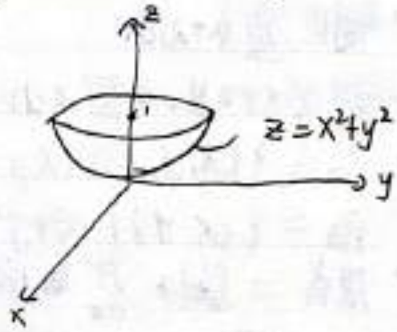


1. 铅直投影法

$$\Omega:\begin{cases}(x,y)\in D_{xy}\\ \varphi_1(x,y)\le z\le\varphi_2(x,y)\end{cases}$$

$\iiint\limits_{\Omega} f(x,y,z)\,dv$

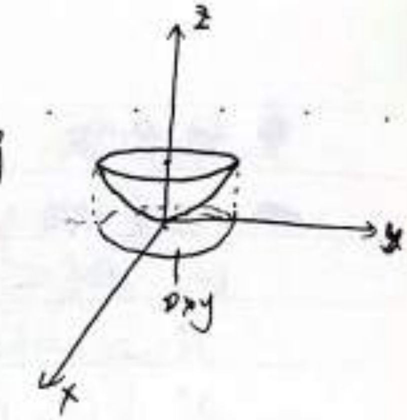
$= \iint\limits_{D_{xy}} dxdy\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz$.



例1. $\iiint\limits_{\Omega}(x+z)\,dV$

Ω 由 $z=x^2+y^2$ 与 $z=1$ 围成

解: $1°$. $\iiint\limits_{\Omega}(x+z)dV = \iiint\limits_{\Omega} z\,dv$



$2°$ $\Omega=\{(x,y,z)\mid (x,y)\in D_{xy},\ x^2+y^2\le z\le 1\}$

$D_{xy}=\{(x,y)\mid x^2+y^2\le 1\}$

$3°$ 原式 $=\iint_{D_{xy}} dxdy\int_{x^2+y^2}^{1} z\,dz$

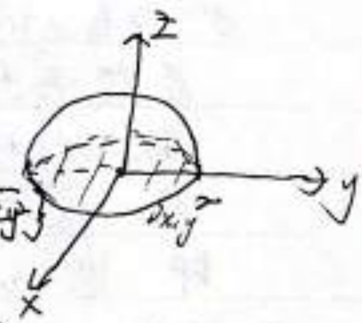
$=\frac{1}{2}\iint_{D_{xy}}[1-(x^2+y^2)^2]\,dxdy$

$=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^1 r(1-r^4)\,dr$

$=\pi\int_0^1(r-r^5)\,dr$

$=\pi\left(\frac{1}{2}-\frac{1}{6}\right)$

$=\frac{\pi}{3}$



例2. $\iiint_\Omega\sqrt{x^2+y^2}\,dV$

解：$1°$ $\Omega=\{(x,y,z)\mid (x,y)\in D_{xy},\ 0\le z\le\sqrt{4-x^2-y^2}\}$

$D_{xy}=\{(x,y)\mid x^2+y^2\le 4\}$

$2°$ $\iiint_\Omega\sqrt{x^2+y^2}\,dV=\iint_{D_{xy}}\sqrt{x^2+y^2}\,dxdy\int_0^{\sqrt{4-x^2-y^2}}dz$

$=\iint_{D_{xy}}\sqrt{x^2+y^2}\cdot\sqrt{4-x^2-y^2}\,dxdy$

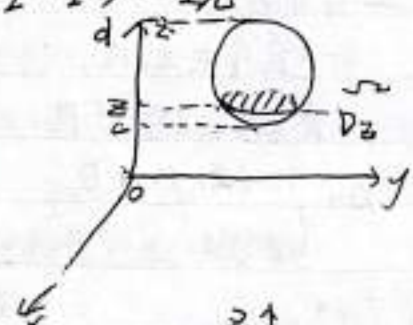
$=\int_0^{2\pi}d\theta\int_0^2 r^2\sqrt{4-r^2}\,dr$

$\overset{r=2\sin t}{=\!=}2\pi\int_0^{\frac{\pi}{2}}4\sin^2t\cdot 2\cos t\cdot 2\cos t\,dt$

$=32\pi\int_0^{\frac{\pi}{2}}\sin^2t(1-\sin^2t)\,dt$

$=32\pi\left(\frac{1}{2}\times\frac{\pi}{2}-\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right)=2\pi^2$

2. 切片法.



$\Omega:\begin{cases}(x,y)\in D_z\\ c\le z\le d\end{cases}$

$\iiint_\Omega f(x,y,z)\,dv=\int_c^d dz\iint_{D_z}f(x,y,z)\,dxdy$

例1. $\iiint_\Omega(x+z)\,dv$

解 $1°$ $\iiint_\Omega(x+z)\,dv=\iiint_\Omega z\,dv$

$2°$ $\Omega=\{(x,y,z)\mid (x,y)\in D_z,\ 0\le z\le 1\}$.

$D_z=\{(x,y)\mid x^2+y^2\le z\}$

$3°$ 原式 $=\int_0^1 z\,dz\iint_{D_z}1\,dxdy=\pi\int_0^1 z^2\,dz$

$=\frac{\pi}{3}$

(二) 球面坐标变换法.

1° 特征:

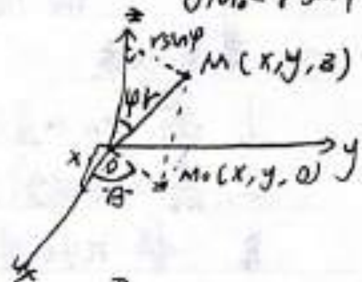
① Ω 边界含 $x^2+y^2+z^2$.

② $f(x,y,z)$ 中含 $x^2+y^2+z^2$.

(θ,φ,r) — 球坐标

$OM=r$

$OM_0=r\sin\varphi$

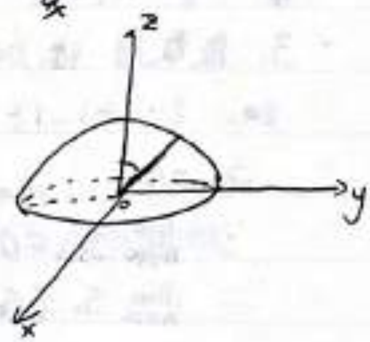


2° 变换:

$$\begin{cases} x=r\cos\theta\sin\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\varphi \end{cases}$$

3°. $dv=r^2\sin\varphi\,dr\,d\theta\,d\varphi$

例1. $\iiint_\Omega \sqrt{x^2+y^2}\,dv$



解: 1 $\begin{cases} x=r\cos\theta\sin\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\varphi \end{cases}$

$(0\le\theta\le2\pi,\ 0\le\varphi\le\frac{\pi}{2},\ 0\le r\le2)$

原式 $=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^2 r^2\sin\varphi\cdot r\sin\varphi\,dr$

$=2\pi\int_0^{\frac{\pi}{2}}\sin^2\varphi\,d\varphi\int_0^2 r^3\,dr=2\pi\times\frac{1}{2}\times\frac{\pi}{2}\times4=2\pi^2$

第九章 级数

$\begin{cases} \left.\begin{matrix}\text{常数项级数} \\ \text{幂级数}\end{matrix}\right\}(一) \\ \text{Fourier 级数}\quad(二) \end{cases}$

Part 1 常数项级数

一、defs.

研究无限从有限开始

研究反常从正常开始

1. $\{a_n\}_{n=1}^{\infty}$: 称 $\sum_{n=1}^{\infty}a_n$ 为常数项级数.

2. 对 $\sum_{n=1}^{\infty}a_n$. $S_n=a_1+a_2+\cdots+a_n$ — 部分和

Notes: ① S_n 与 $\sum_{n=1}^{\infty}a_n$ 不同

② $\lim_{n\to\infty}S_n$ 与 $\sum_{n=1}^{\infty}a_n$ 相同

If $\lim_{n\to\infty}S_n=S$, 则 $\sum_{n=1}^{\infty}a_n=S$;

If $\lim_{n\to\infty}S_n$ 不存在, 称 $\sum_{n=1}^{\infty}a_n$ 发散.

例1. 判断 $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ 收敛性

解: $S_n=\frac{1}{1\times2}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$.

$\because \lim_{n\to\infty}S_n=1$

$\therefore \sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$.

二、基本性质:

1. $\sum_{n=1}^{\infty}a_n=A$. $\sum_{n=1}^{\infty}b_n=B \Rightarrow \sum_{n=1}^{\infty}(a_n\pm b_n)=A\pm B$

2. $\sum_{n=1}^{\infty}a_n=S \Rightarrow \sum_{n=1}^{\infty}ka_n=kS$.

注. $k\neq0$ 时. $\sum_{n=1}^{\infty}a_n$ 与 $\sum_{n=1}^{\infty}ka_n$ 敛散性相同

3. 级数前添加、减少、改变有限项，敛散性不变.

如: $1-1+1-1+1-1+\cdots$

$S_{2n}=0$ $\quad S_{2n+1}=1$

$\because \lim_{n\to\infty}S_{2n}=0\neq\lim_{n\to\infty}S_{2n+1}=1$

$\therefore \lim_{n\to\infty}S_n$ 不存在

$\therefore 1-1+1-1+1-1+\cdots$ 发散

而 $(1-1)+(1-1)+(1-1)+\cdots$

即 $0+0+\cdots=0$.

4. 级数添加括号提高收敛性.

例2. 已知 $\sum_{n=1}^{\infty}a_n$ 收敛. 问 $\sum_{n=1}^{\infty}(a_{2n-1}+a_{2n})$?

解: $\sum_{n=1}^{\infty}(a_{2n-1}+a_{2n})=(a_1+a_2)+(a_3+a_4)+\cdots$ 收敛

5.(必要条件). 若 $\sum_{n=1}^{\infty}a_n$ 收敛. 则 $\lim_{n\to\infty}\overset{\text{一般项}}{a_n}=0$. 反之不对.

证: $S_n=a_1+\cdots+a_n$

$\because \sum_{n=1}^{\infty}a_n$ 收敛. $\therefore \lim_{n\to\infty}S_n\ \exists$

令 $\lim_{n\to\infty}S_n=S$.

$\because a_n=S_n-S_{n-1}$

$\therefore \lim_{n\to\infty}a_n=\lim_{n\to\infty}S_n-\lim_{n\to\infty}S_{n-1}=S-S=0$

三、两个：

1. P-级数 — 形如 $\sum_{n=1}^{\infty}\frac{1}{n^p}$ 称为P-级数

$$\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}收敛, & p>1\\ 发散 & p\le 1\end{cases}$$

Notes：$p=1$时，$\sum_{n=1}^{\infty}\frac{1}{n}$ 称为调和级数。

（$\sum_{n=1}^{\infty}\frac{1}{n}$ 中 $\lim_{n\to\infty}\frac{1}{n}=0$，但 $\sum_{n=1}^{\infty}\frac{1}{n}$ 发散）

2. 几何级数 — $\sum_{n=1}^{\infty}aq^n$ （$a\ne 0$）称为几何级数。

$$\sum_{n=1}^{\infty}aq^n\begin{cases}发散 & ,|q|\ge 1\\ \frac{第一项}{1-q} & ,|q|<1\end{cases}$$

如：$\sum_{n=1}^{\infty}2\left(\frac{2}{3}\right)^n=\frac{2\times\frac{2}{3}}{1-\frac{2}{3}}=4$.

四、正项级数敛散性

（一）def — 称 $\sum_{n=1}^{\infty}a_n$ （$a_n\ge 0,n=1,2,\cdots$）为正项级数。

Notes：

① $S_1\le S_2\le S_3\le\cdots$ 即 $\{S_n\}\uparrow$；

② $\begin{cases}S_n无上界 \Rightarrow \lim_{n\to\infty}S_n=+\infty\,;\ (\sum_{n=1}^{\infty}\frac{1}{n}=+\infty)\\ S_n\le M \Rightarrow \lim_{n\to\infty}S_n\ \exists.\ 即\sum_{n=1}^{\infty}a_n收敛.\end{cases}$

（二）审敛法。

方法一：比较法

Th1.（基本形式） $a_n\ge 0$，$b_n\ge 0$.

① $a_n\le b_n$ 且 $\sum_{n=1}^{\infty}b_n$ 收敛，则 $\sum_{n=1}^{\infty}a_n$ 收敛。

② $a_n\ge b_n$ 且 $\sum_{n=1}^{\infty}b_n$ 发散，则 $\sum_{n=1}^{\infty}a_n$ 发散。

例1. $\sum_{n=1}^{\infty}\sin\frac{\pi}{2^n}$?

解：$\because x\ge 0$ 时，$\sin x\le x$

$\therefore 0<\sin\frac{\pi}{2^n}\le\frac{\pi}{2^n}$

$\because \sum_{n=1}^{\infty}\frac{\pi}{2^n}$ 收敛

$\therefore \sum_{n=1}^{\infty}\sin\frac{\pi}{2^n}$ 收敛

例2. $a_n \le b_n \le c_n$ 且 $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} c_n$ 收敛.

证: $\sum_{n=1}^{\infty} b_n$ 收敛

证: $a_n \le b_n \le c_n \Rightarrow 0 \le b_n - a_n \le c_n - a_n$

$\because \sum_{n=1}^{\infty}(c_n - a_n)$ 收敛. $\therefore \sum_{n=1}^{\infty}(b_n - a_n)$ 收敛

又$\because \sum_{n=1}^{\infty} a_n$ 收敛.

$\therefore \sum_{n=1}^{\infty} b_n$ 收敛.

Th1'(极限形式) $a_n > 0$, $b_n > 0$, 且 $\lim_{n\to\infty} \frac{b_n}{a_n} = l$ $(0 < l < +\infty)$.

则 $\sum_{n=1}^{\infty} a_n$ 与 $\sum_{n=1}^{\infty} b_n$ 敛散性相同.

例3. 判断 $\sum_{n=1}^{\infty} [\frac{1}{n} - \ln(1+\frac{1}{n})]$?

解. $\because x > 0$ 时. $\ln(1+x) < x$. $\therefore \frac{1}{n} - \ln(1+\frac{1}{n}) > 0$

$\because \lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x\to 0} \frac{1 - \frac{1}{1+x}}{2x} = \frac{1}{2}$

$\therefore \lim_{n\to\infty} \frac{\frac{1}{n} - \ln(1+\frac{1}{n})}{\frac{1}{n^2}} = \frac{1}{2}$.

而 $\sum_{n=1}^{\infty} \frac{1}{n^2}$ 收敛. $\therefore \sum_{n=1}^{\infty} [\frac{1}{n} - \ln(1+\frac{1}{n})]$ 收敛

方法二: 比值法: (含阶乘! 用比值法)

Th2. $a_n > 0$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ (≥ 0)

① $\rho < 1$ 时, $\sum_{n=1}^{\infty} a_n$ 收敛;

② $\rho > 1$ 时, $\sum_{n=1}^{\infty} a_n$ 发散;

③ $\rho = 1$?

例4. $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$?

解. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1} \cdot (n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{2^n n!} = 2\lim_{n\to\infty} (\frac{n}{n+1})^n$

$= 2\lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{2}{e} = \rho < 1$

$\therefore$ 级数收敛.

方法三: 根值法.

Th3. $a_n > 0$. $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$

① $\rho < 1$ 时 $\sum_{n=1}^{\infty} a_n$ 收敛

② $\rho > 1$ 时 发散

③ $\rho = 1$?

例5. $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$?

解：$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}=\frac{1}{e}=\rho<1$

∴级数收敛

五. 交错级数及审敛法.

$x=-1\neq 0$

(一) def — $\begin{cases}\sum_{n=1}^{\infty}(-1)^{n-1}a_n=a_1-a_2+a_3-a_4+\cdots \checkmark\\ \sum_{n=1}^{\infty}(-1)^n a_n=-a_1+a_2-a_3+a_4+\cdots\end{cases}$

且 $a_n>0$ $(n=1,2,\cdots)$ 称为交错级数.

(二) 审敛法.

Th (Leibniz 法). 设 $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ $(a_n>0, n=1,2,\cdots)$. 若:

① $\{a_n\}\downarrow$；② $\lim_{n\to\infty}a_n=0$. 则 $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ 收敛. 且 $S\le a_1$.

例6. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$?

解：$a_n=\frac{1}{\sqrt{n}}$

∵ $\{\frac{1}{\sqrt{n}}\}\downarrow$ 且 $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$.

∴ $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ 收敛.

Q1. $\sum_{n=1}^{\infty}a_n$ 收敛，$\sum_{n=1}^{\infty}a_n^2$?　不一定. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ 收敛. 但 $\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{\sqrt{n}}\right]^2$ 发散

Q2. $\sum_{n=1}^{\infty}a_n(a_n\ge 0)$ 收敛 $\sum_{n=1}^{\infty}a_n^2$? ✓ 收敛

例7. $a_n>0$. $\{a_n\}\downarrow$. $\sum_{n=1}^{\infty}(-1)^n a_n$ 发散. 问 $\sum_{n=1}^{\infty}\left(\frac{1}{1+a_n}\right)^n$?

解：1° $\{a_n\}\downarrow$ 且 $a_n>0 \Rightarrow \lim_{n\to\infty}a_n$ ∃

令 $\lim_{n\to\infty}a_n=A$ (≥ 0)

2° $\sum_{n=1}^{\infty}(-1)^n a_n$ 发散 $\Longrightarrow A>0$.

3° ∵ $\lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{1+a_n}\right)^n}=\frac{1}{1+A}=\rho<1$. ∴收敛.

六. 两个概念.

1. 条件收敛 — $\sum_{n=1}^{\infty}a_n$ 收敛. 而 $\sum_{n=1}^{\infty}|a_n|$ 发散

称 $\sum_{n=1}^{\infty}a_n$ 条件收敛

如：$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ 收敛 而 $1+\frac{1}{2}+\frac{1}{3}+\cdots$ 发散

2. 绝对收敛 — 若 $\sum_{n=1}^{\infty}|a_n|$ 收敛 称 $\sum_{n=1}^{\infty}a_n$ 绝对收敛.

Part Ⅱ 幂级数

一、defs.

1. $\begin{cases} \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \\ \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \end{cases}$ 称为幂级数

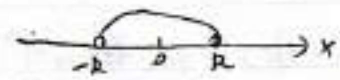
如：$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$

取 $x = \frac{1}{2}$. $\sum_{n=0}^{\infty} (\frac{1}{2})^n$ 收敛. $x = \frac{1}{2}$ — 收敛点

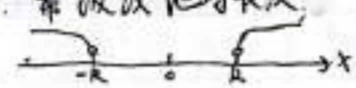
取 $x = 3$ $\sum_{n=0}^{\infty} 3^n$ 发散. $x = 3$ — 发散点

2. (Abel). $\sum_{n=0}^{\infty} a_n x^n$

$\exists R \geq 0$



① $|x| < R$ 或 $-R < x < R$. 幂级数绝对收敛.



② $|x| > R$. 发散.

③ $|x| = R$ 即 $x = \pm R$?

R — 收敛半径. $(-R, R)$ — 收敛区间

二、收敛半径 R，收敛域

Th. 对 $\sum_{n=0}^{\infty} a_n x^n$

若 $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$ 或 $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$

则 ① $\rho = 0$，则 $R = +\infty$；

② $\rho = +\infty$，则 $R = 0$；

③ $0 < \rho < +\infty$，则 $R = \frac{1}{\rho}$.

例1. $\sum_{n=0}^{\infty} n! x^n$?

解：$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} (n+1) = +\infty \Rightarrow R = 0$. 收敛域 $[0]$

例2. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

解. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 \Rightarrow R = +\infty$

∴ 收敛域 $(-\infty, +\infty)$

例3. $\sum_{n=1}^{\infty} \frac{x^n}{2^n \cdot n}$?

解：$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{2^{n+1}\cdot(n+1)} \Big/ \frac{1}{2^n \cdot n} = \frac{1}{2} \Rightarrow R = 2$.

当 $x = -2$ 时. $\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n \cdot n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ 收敛

当 $x = 2$ 时. $\sum_{n=1}^{\infty} \frac{2^n}{2^n \cdot n} = \sum_{n=1}^{\infty} \frac{1}{n}$. 发散

∴收敛域 $[-2, 2)$

三、幂级数的分析性质

$\sum_{n=0}^{\infty} a_n x^n$. $x \in (-R, R)$
$\parallel$
$S(x)$
和函数.

Th1 (逐项可导性) 对 $\sum_{n=0}^{\infty} a_n x^n$ 当 $x \in (-R, R)$ 时.

$(\sum_{n=0}^{\infty} a_n x^n)' = s'(x) = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ 且 $\sum_{n=1}^{\infty} n a_n x^{n-1}$ 收敛半径为 R

Th2 (逐项可积性) ...

$\int_0^x (\sum_{n=0}^{\infty} a_n x^n) dx = \int_0^x s(x) dx = \sum_{n=1}^{\infty} \int_0^x a_n x^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$.

且 $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ 收敛半径 R.

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (-1 < x < 1)$

求和

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad (-1 < x < 1)$. 展开

四、函数展成幂级数.

(一) 直接法 (公式法).

上册: Taylor 公式

设 $f(x)$ 在 $x = x_0$ 邻域内直到 $(n+1)$ 阶可导. 则.

$f(x) = P_n(x) + R_n(x)$

其中 $P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

$R_n(x) \begin{cases} \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} & \text{— Lagrange 型} \\ o((x - x_0)^n) & \text{— 皮亚诺型.} \end{cases}$

Th. 设 $f(x)$ 在 $x = x_0$ 邻域内任意阶可导. 则

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = f(x_0) + f'(x_0)(x - x_0) + \cdots$

(···) - 收敛域和定义域之交

若 $x_0 = 0$.

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$

$f(x)$ 的麦克劳林级数

记: ① $e^x = 1 + x + \frac{x^2}{2!} + \cdots \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty < x < +\infty)$

② $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad (-\infty < x < +\infty)$

③ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (-\infty < x < +\infty)$

④ $\frac{1}{1-x}=1+x+x^2+\cdots=\sum_{n=0}^{\infty}x^n \quad (-1<x<1)$

⑤ $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \quad (-1<x<1)$

⑥ $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} \quad (-1<x\le 1)$

Notes: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\ln 2$

☆ ⑦ $-\ln(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots=\sum_{n=1}^{\infty}\frac{x^n}{n} \quad (-1\le x<1)$

(二) 间接法

工具：$\begin{cases} ①\sim⑦ \\ \text{Th1, Th2.} \end{cases}$

例1. $f(x)=\frac{1}{x^2-1}$ 展成 $(x-2)$ 的幂级数.

解: $f(x)=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$

$\frac{1}{x-1}=\frac{1}{1+(x-2)}=\sum_{n=0}^{\infty}(-1)^n(x-2)^n \quad (1<x<3)$;

$\frac{1}{x+1}=\frac{1}{3+(x-2)}=\frac{1}{3}\cdot\frac{1}{1+\frac{x-2}{3}}=\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-2}{3}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-2)^n \quad (-1<x<5)$

$\therefore f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2}\left(1-\frac{1}{3^{n+1}}\right)(x-2)^n \quad (1<x<3)$

例2. $f(x)=\frac{5x-1}{x^2-x-2}$ 展开成 $(x-1)$ 的幂级数.

解: $f(x)=\frac{5x-1}{(x+1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-2}$

由 $A(x-2)+B(x+1)=5x-1 \Rightarrow \begin{cases} A+B=5 \\ -2A+B=-1 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=3 \end{cases}$

$f(x)=2\cdot\frac{1}{x+1}+3\cdot\frac{1}{x-2}$

$\frac{1}{x+1}=\frac{1}{2+(x-1)}=\frac{1}{2}\cdot\frac{1}{1+\frac{x-1}{2}}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-1}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-1)^n \quad (-1<x<3)$

$\frac{1}{x-2}=\frac{1}{-1+(x-1)}=-\frac{1}{1-(x-1)}=-\sum_{n=0}^{\infty}(x-1)^n \quad (0<x<2)$

$\therefore f(x)=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^n}-3\right](x-1)^n \quad (0<x<2)$

五 求 $S(x)$

工具：$\begin{cases} ①\sim⑦ \\ \text{Th1, Th2.} \end{cases}$

(一) $\sum_{n=0}^{\infty}P(n)x^n$

工具: Th1、④、⑤ $\sum_{n=0}^{\infty}x^n=\frac{1}{1-x} \quad (-1<x<1)$

$\sum_{n=0}^{\infty}(-1)^n x^n=\frac{1}{1+x} \quad (-1<x<1)$

例1. $\sum_{n=1}^{\infty} n x^{n+1}$，求 $S(x)$

解. 1° $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow R=1$

$x=\pm1$ 时 $n\cdot(\pm1)^{n+1} \nrightarrow 0 \ (n\to\infty)$

∴ 收敛域 $(-1,1)$

2°. $S(x) = \sum_{n=1}^{\infty} n x^{n+1} = x^2 \sum_{n=1}^{\infty} n x^{n-1} = x^2 \sum_{n=1}^{\infty} (x^n)'$

$= x^2 \underbrace{\left(\sum_{n=1}^{\infty} x^n\right)'}_{\text{几何级数}} = x^2 \left(\frac{x}{1-x}\right)'$

例2. $\sum_{n=0}^{\infty} n^2 x^n$，求 $S(x)$.

解. 1° $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow R=1$

$x=\pm1$ 时 $n^2(\pm1)^n \nrightarrow 0 \ (n\to\infty)$

∴ 收敛域 $(-1,1)$

2° $S(x) = \sum_{n=0}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} [n(n-1)+n] x^n$

$= \sum_{n=2}^{\infty} n(n-1) x^n + \sum_{n=1}^{\infty} n x^n = x^2 \sum_{n=2}^{\infty} n(n-1) x^{n-2} + x \sum_{n=1}^{\infty} n x^{n-1}$

$= x^2 \sum_{n=2}^{\infty} (x^n)'' + x \sum_{n=1}^{\infty} (x^n)'$

$= x^2 \left(\sum_{n=2}^{\infty} x^n\right)'' + x \left(\sum_{n=1}^{\infty} x^n\right)'$

$= x^2 \left(\frac{x^2}{1-x}\right)'' + x \left(\frac{x}{1-x}\right)'$

(二) $\sum \frac{x^n}{p(n)}$ $\begin{cases} ⑥ \cdot ⑦ \\ \text{消灭分母} \end{cases}$

例1. $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$，求 $S(x)$

解. 1° $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow R=1$

当 $x=\pm1$ 时 $\sum_{n=1}^{\infty} \left|\frac{x(\pm1)^n}{n(n+1)}\right| = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

∴ 收敛域 $[-1,1]$.

2° $S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{x^n}{n} - \sum_{n=1}^{\infty} \frac{x^n}{n+1}$.

① $S(0) = 0$

② $x \neq 0$ 时 $S(x) = -\ln(1-x) - \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1}$

$= -\ln(1-x) - \frac{1}{x} \sum_{n=2}^{\infty} \frac{x^n}{n}$

$= -\ln(1-x) - \frac{1}{x} \left(\sum_{n=1}^{\infty} \frac{x^n}{n} - x\right)$

$= -\ln(1-x) - \frac{1}{x} [-\ln(1-x) - x]$

$=(\frac{1}{x}-1)\ln(1-x)+1 \quad (-1\le x<1$ 且 $x\neq 0)$

② $S(1)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$

$$\therefore S(x)=\begin{cases}0, & x=0\\ 1, & x=1\\ (\frac{1}{x}-1)\ln(1-x)+1, & -1\le x<1 \text{ 且 } x\neq 0\end{cases}$$

10年考过. 例2. $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n}$ 求 $S(x)$

解 1° $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1 \Rightarrow R=1$

$x=\pm1$ 时 $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ 收敛.

2°. $S(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n}$

① $S(0)=1$;

② $xS(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}=\sum_{n=0}^{\infty}(-1)^n\int_0^x x^{2n}dx=\sum_{n=0}^{\infty}\int_0^x(-1)^n x^{2n}dx$

$=\int_0^x(\sum_{n=0}^{\infty}(-1)^n x^{2n})dx=\int_0^x\frac{1}{1+x^2}dx=\arctan x.$

$\therefore S(x)=\frac{\arctan x}{x}$

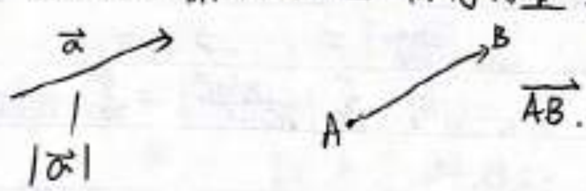
$$S(x)=\begin{cases}1, & x=0\\ \frac{\arctan x}{x}, & -1\le x\le 1 \text{ 且 } x\neq 0\end{cases}$$

第十章 向量代数与空间解析几何(一)

Part I 工具 - 向量理论

一、def.

1. 向量 — 有大小,有方向的量 称为向量.



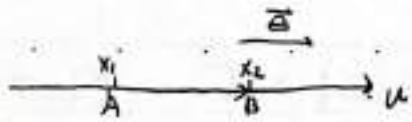
If $|\vec{\alpha}|=0$, $\vec{\alpha}=\vec{0}$;

If $|\vec{\alpha}|=1$, 称 $\vec{\alpha}$ 为单位向量

2. 向量的坐标

1-dim. ruler

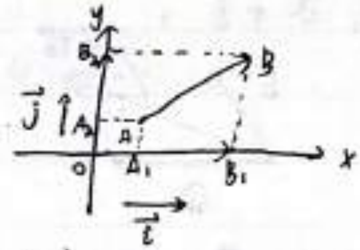
$\overrightarrow{AB}=(4-1)\vec{e}=3\vec{e}$ ($\overrightarrow{AB}$ 的坐标) $\quad \overrightarrow{AC}=(-3-1)\vec{e}=-4\vec{e}$ ($\overrightarrow{AC}$ 的坐标)



$$\overrightarrow{AB} = (x_2 - x_1)\vec{e}$$

2-dim：

$A(x_1, y_1)$. $B(x_2, y_2)$.

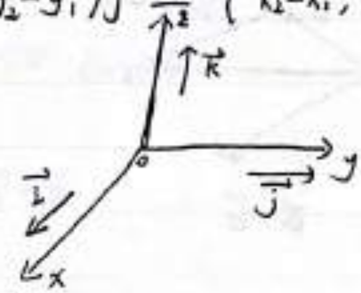


$$\overrightarrow{AB} = \overrightarrow{A_1B_1} + \overrightarrow{A_2B_2}$$

$$\overrightarrow{A_1B_1} = (x_2 - x_1)\vec{i}, \quad \overrightarrow{A_2B_2} = (y_2 - y_1)\vec{j}$$

$$\overrightarrow{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} \triangleq \{x_2 - x_1, y_2 - y_1\}$$ — $\overrightarrow{AB}$的坐标形式.

3-dim：

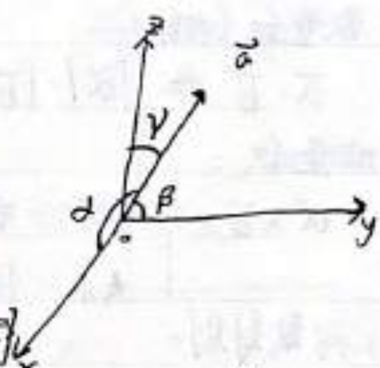


$A(x_1, y_1, z_1)$. $B(x_2, y_2, z_2)$

$$\overrightarrow{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

$$\triangleq \{x_2 - x_1, y_2 - y_1, z_2 - z_1\}.$$

3、方向角与方向余弦



设 $\vec{\alpha} = \{a_1, b_1, c_1\}$

$\vec{\alpha}$与 x, y, z 轴正方向夹角.

设 α, β, γ — $\vec{\alpha}$ 的方向角

1° $\vec{\alpha}^{\circ} = \frac{1}{|\vec{\alpha}|}\vec{\alpha}$ $\quad |\vec{\alpha}| = \sqrt{a_1^2 + b_1^2 + c_1^2}$

$$= \left\{\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}\right\}$$

2° $\cos\alpha = \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$, $\cos\beta = \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$, $\cos\gamma = \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$.

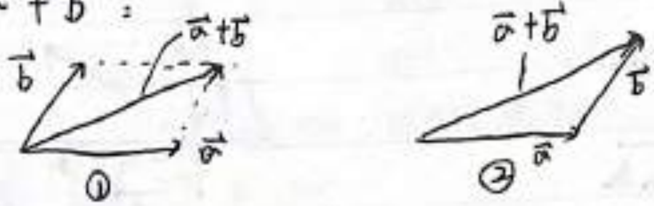
① $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$;

② $\{\cos\alpha, \cos\beta, \cos\gamma\} = \vec{\alpha}^{\circ} = \frac{1}{|\vec{\alpha}|}\vec{\alpha}$

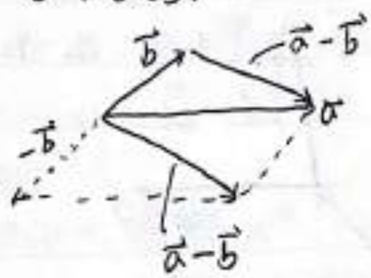
二、向量的运算

(一) 几何规则:

1. $\vec{a}+\vec{b}$:



2. $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$:



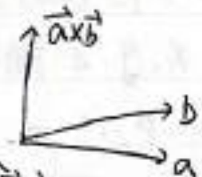
3. $k\vec{a}$
$$\begin{cases} k>0, & k\vec{a} \text{ 方向与 } \vec{a} \text{ 同，长为 } \vec{a} \text{ 的 } k \text{ 倍.} \\ k=0, & k\vec{a}=\vec{0}; \\ k<0, & k\vec{a} \text{ 方向与 } \vec{a} \text{ 反，长为 } \vec{a} \text{ 的 } |k| \text{ 倍} \end{cases}$$

4. 数量积(内积):

$$\vec{a}\cdot\vec{b} \triangleq |\vec{a}|\cdot|\vec{b}|\cdot\cos(\widehat{\vec{a},\vec{b}}).$$

5. 向量积:

$$\vec{a}\times\vec{b}: \begin{cases} \text{方向：右手准则.} \\ \text{大小：} |\vec{a}\times\vec{b}|=|\vec{a}|\cdot|\vec{b}|\sin(\widehat{\vec{a},\vec{b}}) \end{cases}$$



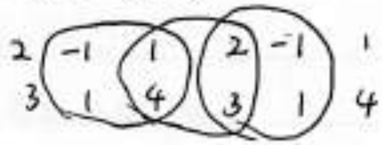
(二) 代数规则:

设 $\vec{a}=\{a_1,b_1,c_1\}$ $\vec{b}=\{a_2,b_2,c_2\}$

1. $\vec{a}+\vec{b}=\{a_1+a_2,\ b_1+b_2,\ c_1+c_2\}$;
2. $\vec{a}-\vec{b}=\{a_1-a_2,\ b_1-b_2,\ c_1-c_2\}$;
3. $k\vec{a}=\{ka_1,\ kb_1,\ kc_1\}$
4. $\vec{a}\cdot\vec{b}=a_1a_2+b_1b_2+c_1c_2$;
5. $\vec{a}\times\vec{b}=\{\quad,\quad,\quad\}$

如: $\vec{a}=\{2,-1,1\}$ $\vec{b}=\{3,1,4\}$.

$\vec{a}\times\vec{b}=\{-5,-5,5\}$



Notes:

1. $\vec{a}\cdot\vec{b}$.

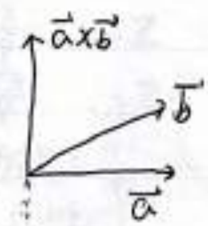
① $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

② $\vec{a}\cdot\vec{a}=|\vec{a}|^2$ $\quad \vec{a}\cdot\vec{a}=0 \Leftrightarrow \vec{a}=\vec{0}$.

③ $\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$

2. $\vec{a}\times\vec{b}$:

① $\vec{a}\times\vec{b}=-\vec{b}\cdot\vec{a}$:

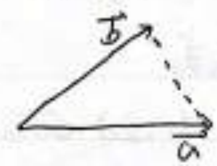


☆ ② $\vec{a}\times\vec{b}\perp\vec{a}$, $\vec{a}\times\vec{b}\perp\vec{b}$;

③ $\vec{a}\times\vec{b}=\vec{0} \Leftrightarrow \vec{a}/\!/\vec{b} \Leftrightarrow \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

④ $|\vec{a}\times\vec{b}|=|\vec{a}|\cdot|\vec{b}|\sin(\vec{a},\vec{b})$

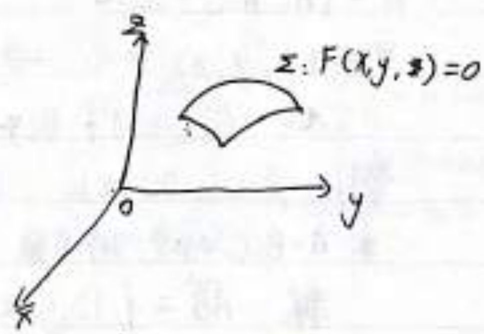
$=2S_{\triangle}$.



Part II 应用

一、空间曲面

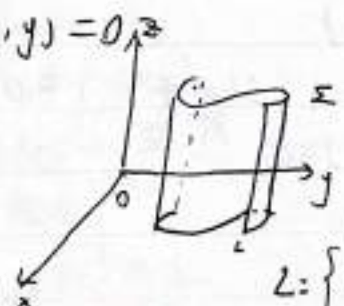
Σ: $F(x,y,z)=0$



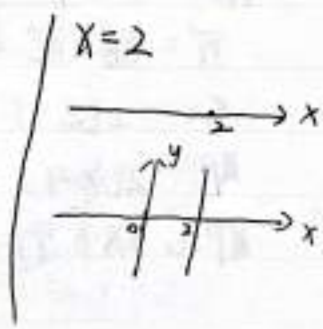
(一) 特殊曲面.

1. 柱面.

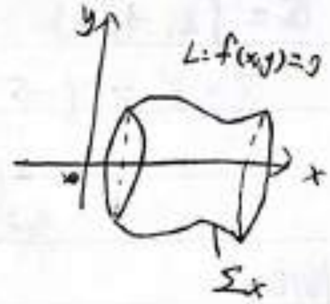
① Σ: $F(x,y)=0$,



$L: \begin{cases} F(x,y)=0 \\ z=0 \end{cases}$



2、旋转曲面.（基础仅讲2-dim）



$$L:\begin{cases} f(x,y)=0 \\ z=0 \end{cases}$$

Σ_x: $f(x,\pm\sqrt{y^2+z^2})=0$

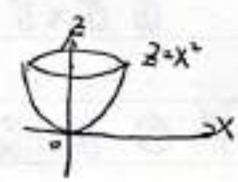
Σ_y: $f(\pm\sqrt{x^2+z^2},y)=0$

例1. $L:\begin{cases} \frac{x^2}{4}+y^2=1 \\ z=0 \end{cases}$ 求 Σ_x,Σ_y

解: Σ_x: $\frac{x^2}{4}+y^2+z^2=1$

Σ_y: $\frac{x^2}{4}+y^2+\frac{z^2}{4}=1$

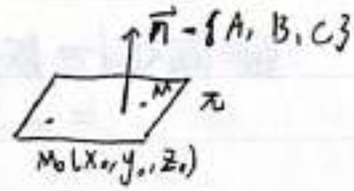
例2. $L:\begin{cases} z=x^2 \\ y=0 \end{cases}$ 求 Σ_z



解: Σ_z: $z=x^2+y^2$

(二) 退化—平面

1. 点法式



$M_0(x_0,y_0,z_0)\in\pi$.

$\vec{n}=\{A,B,C\}\perp\pi$

$\forall M(x,y,z)\in\pi \Rightarrow \vec{n}\perp\overrightarrow{M_0M} \Leftrightarrow \vec{n}\cdot\overrightarrow{M_0M}=0$.

$\therefore\ \pi$: $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$.

例1. $A(1,0,-1)$ $B(2,1,1)$. $C(0,2,1)$

求 A·B.C 确定的平面

解. $\overrightarrow{AB}=\{1,1,2\}$ $\overrightarrow{AC}=\{-1,2,2\}$

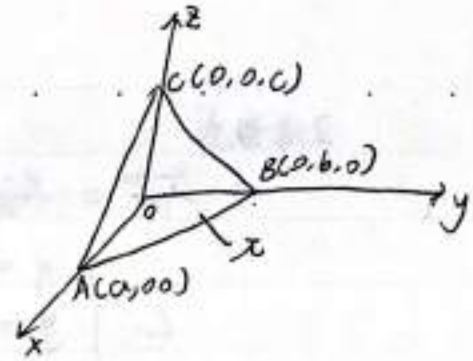
$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\{-2,-4,3\}$

$\therefore\ \pi$: $-2(x-1)-4(y-0)+3(z+1)=0$

即 $2(x-1)+4y-3(z+1)=0$

即π $2x+4y-3z-5=0$

2. 截距式:



$\overrightarrow{AB}=\{-a,b,0\}$

$\overrightarrow{AC}=\{-a,0,c\}$

$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\{bc,ac,ab\}$

$\pi:\quad bc(x-a)+ac(y-0)+ab(z-0)=0$

$\Rightarrow \pi:\ \frac{x-a}{a}+\frac{y}{b}+\frac{z}{c}=0$

$\therefore \pi:\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

3. 一般式

$\pi:\ Ax+By+Cz+D=0$

(三) 空间曲面 —— $\begin{cases}\text{切平面}\\ \text{法线}\end{cases}$

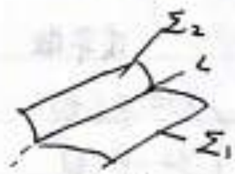
$\Sigma:\ F(x,y,z)=0$

$M_0(x_0,y_0,z_0)\in\Sigma$

$\vec{n}=\{F'_x,F'_y,F'_z\}_{M_0}$

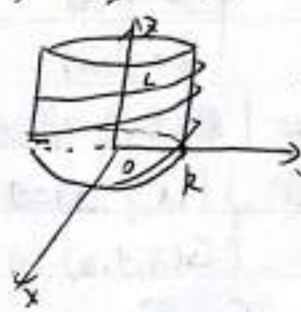
二、空间曲线

(一) 空间曲线表达形式



1. 一般式

$L:\ \begin{cases}F(x,y,z)=0\\ G(x,y,z)=0\end{cases}$

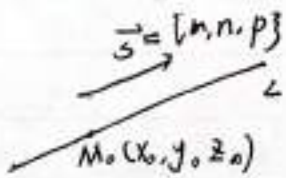


$\begin{cases}x=R\cos\omega t\\ y=R\sin\omega t\\ z=v_0 t\end{cases}$

2. 参数式

$L:\ \begin{cases}x=\varphi(t)\\ y=\psi(t)\\ z=\omega(t)\end{cases}$

(二) 退化 — 直线



1. 点向式

$M_0(x_0,y_0,z_0)\in L.$

$\vec{s}=\{m,n,p\}\,/\!/\,L.$

$\forall M(x,y,z)\in L \Rightarrow \overrightarrow{M_0M}\,/\!/\,\vec{s}\quad \therefore L:\ \frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}$

2. 参数式

令 $\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}=t$.

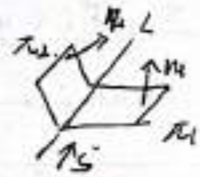
$\therefore$ $L:\begin{cases} x=x_0+mt \\ y=y_0+nt \\ z=z_0+pt \end{cases}$

3. 一般式：

$L:\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$

例1. $L:\begin{cases} x-y-z-4=0 \\ 2x+y-1=0 \end{cases}$ 化为点向式.

解：1° $M_0(1,-1,2)\in L$.



2° $\vec{s}=\vec{n}_1\times\vec{n}_2=\{1,-1,-1\}\times\{2,1,0\}$

$=\{1,-2,3\}$

$\therefore L: \frac{x-1}{1}=\frac{y+1}{-2}=\frac{z-2}{3}$

(三). 空间曲线 $\begin{cases} \text{切线} \\ \text{法平面} \end{cases}$

1. $L:\begin{cases} x=\varphi(t) \\ y=\psi(t) \\ z=\omega(t) \end{cases}$ $t=t_0.\Rightarrow M_0$

$\vec{T}=\{\varphi'(t_0),\psi'(t_0),\omega'(t_0)\}$

2. $L:\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ $M_0(x_0,y_0,z_0)\in L$.

$\vec{T}=\vec{n}_1\times\vec{n}_2$

$=\{F_x',F_y',F_z'\}_{M_0}\times\{G_x',G_y',G_z'\}_{M_0}$

三 距离.

(一) The distance between two points.

$A(x_1, y_1, z_1)$ $\quad B(x_2, y_2, z_2)$.

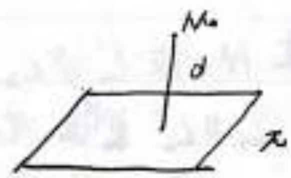
$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

(二) The distance from a point to a place

π: $Ax+By+Cz+D=0$

$M_0(x_0, y_0, z_0) \notin \pi$.

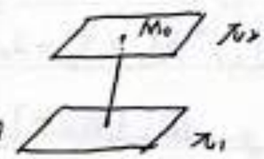
$$d = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$



(三) 平行平面之距

π_1: $Ax+By+Cz+D_1=0$

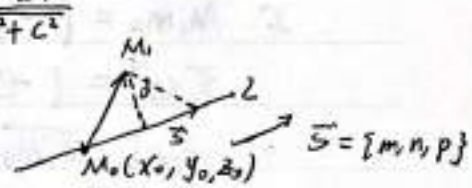
π_2: $Ax+By+Cz+D_2=0$



$\forall M_0(x_0, y_0, z_0) \in \pi_2$ $\quad Ax_0+By_0+Cz_0+D_2=0$

$$d = \frac{|Ax_0+By_0+Cz_0+D_1|}{\sqrt{A^2+B^2+C^2}} = \frac{|D_1-D_2|}{\sqrt{A^2+B^2+C^2}}$$

(四) 点到直线之距



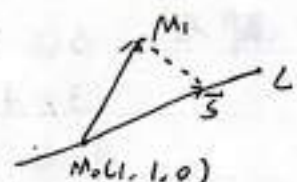
$|\overrightarrow{M_0M_1} \times \vec{s}| = 2S_\Delta$

$|\vec{s}| \cdot d = 2S_\Delta$

$|\vec{s}| \cdot d = |\overrightarrow{M_0M_1} \times \vec{s}|$

例1. 求点 $M_1(1,2,1)$ 到直线 $L: \frac{x-1}{1} = \frac{y-1}{0} = \frac{z}{-1}$ 之距.

解: $M_0(1,1,0) \in L$. $\vec{s} = \{1,0,-1\}$



$\overrightarrow{M_0M_1} = \{0,1,1\}$

$\overrightarrow{M_0M_1} \times \vec{s} = \{-1,1,-1\}$

$|\vec{s}| = \sqrt{2}$ $\quad$ 由 $|\vec{s}|d = |\overrightarrow{M_0M_1} \times \vec{s}|$

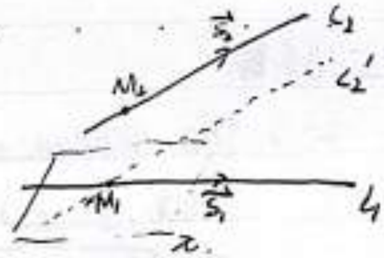
$d = \frac{\sqrt{3}}{\sqrt{2}}$

例2. $L: \begin{cases} x-y-z+1=0 \\ 2x-y+z-2=0 \end{cases}$ $\quad M_1(1,-1,2)$

解: 1° $M_0(1,1,\ \ 1\ \ ) \in L$ $\quad \vec{s} = \{1,-1,-1\} \times \{2,-1,1\} = \{-2,-3,1\}$

2° $\overrightarrow{M_0M_1} = \{0,-2,1\}$ $\quad \overrightarrow{M_0M} \times \vec{s}$

(五) 异面直线之距



Notes:

① L_1, L_2 共面 $\Rightarrow \vec{s_1}\times\vec{s_2} \perp \overrightarrow{M_1M_2}$

$\Leftrightarrow (\vec{s_1}\times\vec{s_2})\cdot\overrightarrow{M_1M_2}=0$

② L_1, L_2 异面 $\Leftrightarrow (\vec{s_1}\times\vec{s_2})\cdot\overrightarrow{M_1M_2}\neq 0$

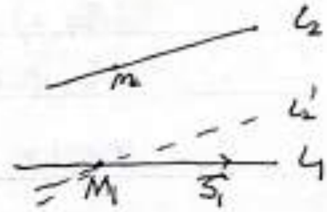
If $(\vec{s_1}\times\vec{s_2})\cdot\overrightarrow{M_1M_2}\neq 0$

过 M_1 作 $L_2' \parallel L_2$

L_1 与 L_2' 成平面 π. $M_1\in\pi$

$\vec{n} = \vec{s_1}\times\vec{s_2} \Rightarrow \pi$.

异面直线之距 等于 M_2 与 π 之距



例. 1° $M_1(9,-2,0)\in L_1$, $\vec{s_1}=\{4,-3,1\}$

$M_2(0,-7,2)\in L_2$, $\vec{s_2}=\{-2,9,2\}$

2° $\overrightarrow{M_1M_2}=\{-9,-5,2\}$

$\vec{s_1}\times\vec{s_2}=\{-15,-10,30\}$

$\because \vec{s_1}\times\vec{s_2}\cdot\overrightarrow{M_1M_2}=135+50+60\neq 0$

$\therefore L_1, L_2$ 异面.

3° 过 M_1 作 L_2' 平行 L_2, L_1 与 L_2' 成平面 π.

$\pi: -15(x-9)-10(y+2)+30(z-0)=0$

即 π: $3(x-9)+2(y+2)-6z=0$

$3x+2y-6z-23=0$

$\therefore d=\dfrac{|3\times 0+2\times(-7)-6\times 2-23|}{\sqrt{9+4+36}}=\dfrac{49}{7}=7$

十一

第十一章 曲线与曲面积分

积分域	积分号	如：
线状	$\int$	① $\int_a^b f(x)dx$ ② $\int_L$
面状	$\iint$	① $\iint_D f(x,y)d\sigma$ ② $\iint_\Sigma$
体状	$\iiint$	$\iiint_\Omega f(x,y,z)dv$

Part I 曲线积分

一、对弧长的曲线积分（第一类曲线积分）

1° 背景

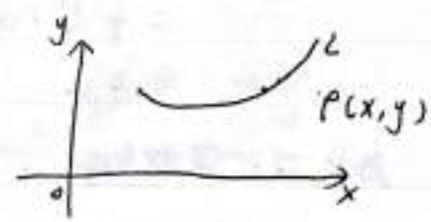
2° 抽象 $\Rightarrow$ 积分种类

3° 性质

4° 计算方法

5° 应用

（一）背景



例1. 求m

1° $\forall ds \subset L$；

2° $dm = \rho(x,y)ds$

3° $m = \int_L dm = \int_L \rho(x,y)ds$.

（二）def — $\int_L f(x,y)ds$

称 $f(x,y)$ 在曲线段L上对弧长的曲线积分.

（三）性质：

1. $\int_L 1ds = l$.

2. ① L关于y轴对称. 右 L_1.

If $f(-x,y) = -f(x,y)$. $\int_L f(x,y)ds = 0$;

If $f(-x,y) = f(x,y)$. $\int_L f(x,y)ds = 2\int_{L_1} f(x,y)ds$.

② L关于x轴对称. 上 L_1

If $f(x,-y) = -f(x,y)$. $\int_L f(x,y)ds = 0$

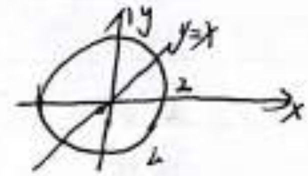
If $f(x,-y) = f(x,y)$. $\int_L f(x,y)ds = 2\int_{L_1} f(x,y)ds$.

③ L关于 $y=x$ 对称. 则 $\int_L f(x,y)ds = \int_L f(y,x)ds$.

(四) 计算法

方法一：替代法

例1. $\int_L (x^2+2y)\,ds$.



解：$I = \int_L (x^2+2y)\,ds$.

$= \int_L x^2 ds = \int_L y^2 ds$.

$= \frac{1}{2}\int_L (x^2+y^2)\,ds = \frac{4}{2}\int_L ds$

$= 2\times 2\pi\times 2 = 8\pi$

例2. $I=\int_L (x-2y)^2 ds$. $L: \frac{x^2}{4}+y^2=1$ 且 L的长为 a.

解：$I=\int_L (x^2+4y^2-4xy)\,ds$

$= \int_L (x^2+4y^2)\,ds$

$= 4\int_L (\frac{x^2}{4}+y^2)\,ds$

$= 4\int_L 1\,ds$

$= 4a$

方法二：定积分法

对 $\int_L f(x,y)\,ds$.

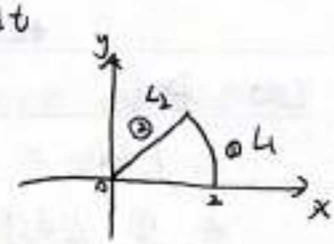
1. L: $y=\varphi(x)$ $(a\le x\le b)$

$$\int_L f(x,y)\,ds = \int_a^b f[x,\varphi(x)]\cdot\sqrt{1+\varphi'^2(x)}\,dx$$

2. L: $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$ $(\alpha\le t\le\beta)$

$$\int_L f(x,y)\,ds = \int_\alpha^\beta f[\varphi(t),\psi(t)]\cdot\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$$

例3. 求 $\int_L x e^{x^2+y^2} ds$.



解：$\int_L x e^{x^2+y^2} ds = \int_{L_1} x e^{x^2+y^2} ds + \int_{L_2} x e^{x^2+y^2} ds$.

L_1: $\begin{cases} x=2\cos t \\ y=2\sin t \end{cases}$ $(0\le t\le \frac{\pi}{4})$.

$$\int_{L_1} x e^{x^2+y^2} ds = \int_0^{\frac{\pi}{4}} 2\cos t\; e^4\cdot\sqrt{4\sin^2 t+4\cos^2 t}\,dt$$

$$= 4e^4\int_0^{\frac{\pi}{4}} \cos t\,dt = 2\sqrt{2}e^4$$

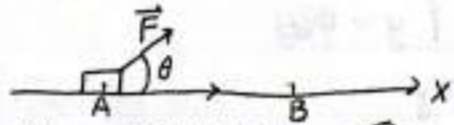
L_2: $y=x$ $(0\le x\le\sqrt{2})$

$$\int_{L_2} x e^{x^2+y^2} ds = \int_0^{\sqrt{2}} x e^{2x^2}\cdot\sqrt{1+1}\,dx = \frac{\sqrt{2}}{4}\; e^{2x^2}\Big|_0^{\sqrt{2}} = \frac{\sqrt{2}}{4}(e^4-1)$$

二、对坐标的曲线积分（第二类曲线积分）

（一）背景：做功

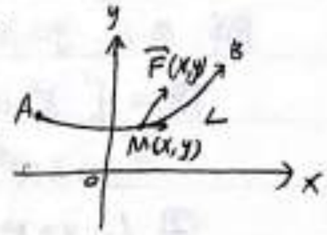
Case 1. 双理想：



$w = |\vec{F}| \cdot \cos\theta \cdot |\overrightarrow{AB}| = |\vec{F}| \cdot |\overrightarrow{AB}| \cdot \cos(\vec{F}, \overrightarrow{AB}) = \underline{\vec{F} \cdot \overrightarrow{AB}}$. 点乘

Case 2. (2-dim) 双不理想：

$\vec{F}(x,y) = \{P(x,y), Q(x,y)\}$　　$w = ?$

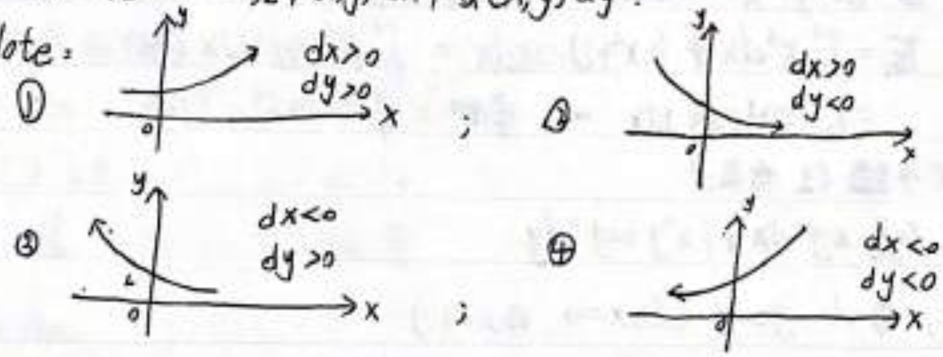


1° $\forall \vec{ds} \subset L$.　$\vec{ds} = \{dx, dy\}$;

2° $dw = \vec{F} \cdot \vec{ds} = P(x,y)dx + Q(x,y)dy$;

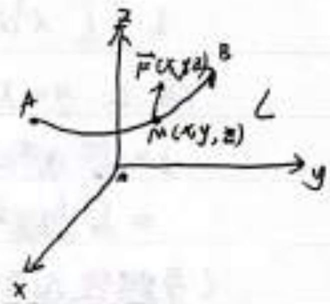
3° $w = \int_L dw = \int_L P(x,y)\,dx + Q(x,y)\,dy$.

Note:



Case 3. (3-dim) 双不理想：

$\vec{F} = \{P(x,y,z), Q(x,y,z), R(x,y,z)\}$.



1° $\forall \vec{ds} \subset L$.　$\vec{ds} = \{dx, dy, dz\}$;

2° $dw = \vec{F} \cdot \vec{ds} = Pdx + Qdy + Rdz$.

3° $w = \int_L dw = \underline{\int_L Pdx + Qdy + Rdz}$

（二）def:

1. (2-dim): $\int_L Pdx + Qdy$

$\int_L Pdx$ — $P(x,y)$ 在有向曲线段 L 上对坐标的曲线积分.

2. (3-dim): $\int_L Pdx + Qdy + Rdz$.

（三）性质

1. $\int_{L^-} = -\int_L$

（四）$\int_L Pdx + Qdy$ 计算方法.

方法一：定积分法.

Case 1: $L: y = \varphi(x)$ （起点 $x = a$，终点 $x = b$）.

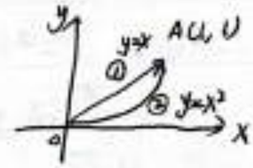
$$\int_L Pdx + Qdy = \int_a^b P[x, \varphi(x)]\,dx + Q[x, \varphi(x)]\,\varphi'(x)\,dx.$$

Case 2. $L: \begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$ （起点 $t=\alpha$，终点 $t=\beta$）.

$\int_L P\,dx + Q\,dy$

$= \int_\alpha^\beta P[\varphi(t),\psi(t)]\varphi'(t)dt + Q[\varphi(t),\psi(t)]\psi'(t)dt.$

例1. $\int_L xy\,dx + (2y+1)dy$



解：① $L: y=x$ （起点 $x=0$，终点 $x=1$）

原 $= \int_0^1 x^2dx + (2x+1)dx = \int_0^1 (x+1)^2 d(x+1)$

$= \frac{1}{3}(x+1)^3\Big|_0^1 = \frac{7}{3}$

② $L: y=x^2$ （起点 $x=0$，终点 $x=1$）

原 $= \int_0^1 x^3dx + (2x^2+1)\cdot 2x\,dx = \int_0^1 (x^3+4x^3+2x)dx$

$= \int_0^1 (5x^3+2x)dx = \frac{5}{4}+1 = \frac{9}{4}$

（与路径有关）

例2. $I=\int_L xy^2\,dx + (x^2y+2y)dy$

解：① $L: y=x$ （起 $x=0$，终 $x=1$）

$I = \int_0^1 x^3dx + (x^3+2x)dx = \int_0^1 (2x^3+2x)dx = \frac{1}{2}+1 = \frac{3}{2}$

② $L: y=x^2$ （起 $x=0$，终 $x=1$）

$I = \int_0^1 x^5dx + \int (x^4+2x^2)2x\,dx$

$= \int_0^1 (3x^5+4x^3)dx = \frac{1}{2}+1 = \frac{3}{2}$

（与路径无关）

例3. $I=\int_L x\,dy - 2y\,dx$

解：1° $L: \begin{cases} x=2\cos t \\ y=2\sin t \end{cases}$ （起 $t=\pi$，终 $t=0$）

2° $I = \int_\pi^0 2\cos t\cdot 2\cos t\,dt - 2\cdot 2\sin t\cdot(-2\sin t)\,dt$

$= -\int_0^\pi (4\cos^2 t + 8\sin^2 t)\,dt$

$= -4\int_0^\pi (1+\sin^2 t)dt$

$= -4(\pi + 2I_2) = -4(\pi + 2\times\frac{1}{2}\times\frac{\pi}{2}) = -6\pi$

方法二：二重积分法（Green 公式）.

单连通区域

逆正，顺负

多连通区域

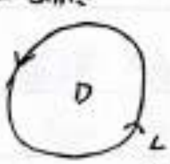
外逆，内顺为正方向

域与界的关系

1-dim: $F(b)-F(a)=\int_a^b f(x)dx$

$[a,b]$ 域　　　a, b 边界

2-dim:



D—域 —— 二重积分

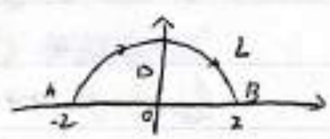
L—边界 —— 曲线积分

Th (Green) 若 ① D—连通区域，L为D的正向边界；

② $P(x,y)$, $Q(x,y)$ 在D上连续可偏导，

则 $\oint_L P\,dx+Q\,dy=\iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma$

例1. $I=\int_L x\,dy-2y\,dx$



解: 1° $P=-2y$, $Q=x$

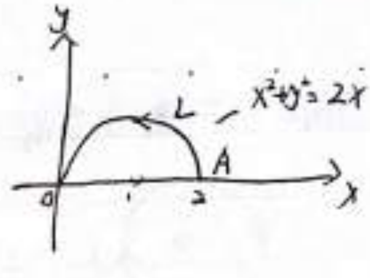
$\frac{\partial Q}{\partial x}=1$　　$\frac{\partial P}{\partial y}=-2$

2° $I=\int_L x\,dy-2y\,dx=\oint_{L+\overline{BA}} x\,dy-2y\,dx+\int_{\overline{AB}} x\,dy-2y\,dx$

3° $\oint_{L+\overline{BA}} x\,dy-2y\,dx=-\iint_D 3\,d\sigma=-3\times2\pi=-6\pi$

$\int_{\overline{AB}} x\,dy-2y\,dx=\int_{-2}^{2}-2\times0\,dx=0$

$\therefore I=-6\pi$



例2. $\int_L (x+2y)dx + x^3dy$.

解: 1° $P=x+2y \quad Q=x^3$

$\frac{\partial Q}{\partial x}=3x^2, \quad \frac{\partial P}{\partial y}=2.$

2° $I=\int_L (x+2y)dx+x^3dy = \oint_{L+\overline{OA}} - \int_{\overline{OA}}$

3° $\oint_{L+\overline{OA}} = \iint_D (3x^2-2)d\sigma = 3\iint_D x^2 d\sigma - 2\times\frac{\pi}{2}$

令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$ $(0\le\theta\le\frac{\pi}{2}, \quad 0\le r\le 2\cos\theta)$

$\oint_{L+\overline{OA}} = 3\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta} r^3\cos^2\theta dr - \pi$

$= 3\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta\int_0^{2\cos\theta} r^3 dr - \pi$

$= 12\int_0^{\frac{\pi}{2}}\cos^6\theta d\theta - \pi$

$= 12\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} - \pi = \frac{7}{8}\pi.$

$\int_{\overline{OA}} = \int_0^2 (x+2\times 0)dx = 2$

$\therefore\ I=\frac{7}{8}\pi-2.$

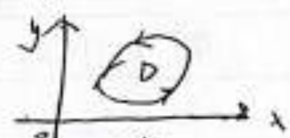
例3. $\oint_L \frac{ydx-xdy}{x^2+y^2}$ L为不过0的正向闭曲线

解. 1° $P=\frac{y}{x^2+y^2} \qquad Q=\frac{-x}{x^2+y^2}$

$\frac{\partial Q}{\partial x} = -\frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\frac{\partial P}{\partial y} = \frac{x^2+y^2-2y\cdot y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \quad ((x,y)\ne(0,0))$



2° Case1 $(\mathring{x},\mathring{y})\notin D$

$I=\oint\frac{ydx-xdy}{x^2+y^2} = \iint_D 0\, d\sigma = 0$

Case2 $(\mathring{x},\mathring{y})\in D$

令L_0: $x^2+y^2=r^2$

$(r>0$, L_0在L内, L_0逆时针$)$

$\oint_{L+L_0^-} = \iint_{D_1} 0\, d\sigma = 0$

$\Rightarrow \oint_L\frac{ydx-xdy}{x^2+y^2} = \int_{L_0}\frac{ydx-xdy}{x^2+y^2} = \frac{1}{r^2}\oint_{L_0} ydx-xdy$

$= \frac{1}{r^2}\iint_{D_2}(-2)d\sigma = -\frac{2}{r^2}\times\pi r^2 = -2\pi$

方法三：曲线积分与路径无关问题

Th. 若 ① D 一单连通区域，

② $P(x,y)$、$Q(x,y)$ 在D上连续可偏导.

以下四个命题等价：

① $\int_L Pdx+Qdy$ 与路径无关；

② 任取闭曲线 $c \subset D$，有 $\oint_c Pdx+Qdy=0$；

√ ③ $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ （柯西黎曼 C-R）

④ $\exists\, u(x,y)$，使

$$du = Pdx+Qdy$$

（即 $\frac{\partial u}{\partial x}=P$，$\frac{\partial u}{\partial y}=Q$）

Notes:

① If $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$，则

$$\int_L Pdx+Qdy=\int_{(x_0,y_0)}^{(x_1,y_1)} Pdx+Qdy$$

$$=\int_{x_0}^{x_1} P(x,y_0)\,dx+\int_{y_0}^{y_1} Q(x_1,y)\,dy$$

② If $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$，且 $Pdx+Qdy=du$

（如：$ydx+xdy=d(xy)$.　$xy^2dx+x^2ydy=d(\frac{1}{2}x^2y^2)$）.

则 $\int_L Pdx+Qdy=\int_{(x_0,y_0)}^{(x_1,y_1)} du=u(x_1,y_1)-u(x_0,y_0)$.

③ If $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$，则

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)} Pdx+Qdy$$

$$=\int_{x_0}^{x} P(x,y_0)\,dx+\int_{y_0}^{y} Q(x,y)\,dy$$

例1. $\varphi(x)$可导，且 $\varphi(0)=2$　$\int_L xy^2dx+\varphi(x)y\,dy$ 与路径无关.

① $\varphi(x)$！　② $\int_{(1,2)}^{(2,3)}\cdots$？

解：① $P=xy^2$，$Q=\varphi(x)y \Rightarrow \varphi'(x)y=2xy \Rightarrow \varphi'(x)=2x$

$\Rightarrow \varphi(x)=x^2+C$

$\because \varphi(0)=2$　$\therefore \varphi(x)=x^2+2$.

② 法一：$I=\int_{(1,2)}^{(2,3)} xy^2dx+(x^2+2)ydy=\int_1^2 4xdx+\int_2^3 6ydy=$ ！

法二：$\because xy^2dx+(x^2+2)ydy=(xy^2dx+x^2ydy)+2ydy$

$=d(\frac{1}{2}x^2y^2)+d(y^2)=d(\frac{1}{2}x^2y^2+y^2)$

$\therefore I=(\frac{1}{2}x^2y^2+y^2)\Big|_{(1,2)}^{(2,3)}$

Part Ⅱ 曲面积分

一、对面积的曲面积分(第一类曲面积分).

(一) 背景: m?

1° $\forall ds \subset \Sigma$;

2° $dm = \rho(x,y,z)\,ds$;

3° $m = \iint_\Sigma dm = \iint_\Sigma \rho(x,y,z)\,ds$.

(二) ~~性质~~. **def—** $\iint_\Sigma f(x,y,z)\,ds$.

$f(x,y,z)$ 在曲面Σ上对面积的曲面积分.

(三) 性质:

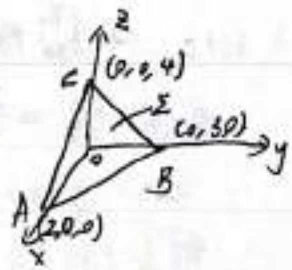
1. $\iint_\Sigma 1\,ds = A$

2. ① Σ关于xoy面对称,上Σ_1.

If $f(x,y,-z) = -f(x,y,z)$. 则 $\iint_\Sigma f(x,y,z)\,ds = 0$;

If $f(x,y,-z) = f(x,y,z)$. 则 $\iint_\Sigma f\,ds = 2\iint_{\Sigma_1} f\,ds$.

(四) 计算方法 — 二重积分法.



例1. $I = \iint_\Sigma (2x + \frac{4}{3}y + z)\,ds$.

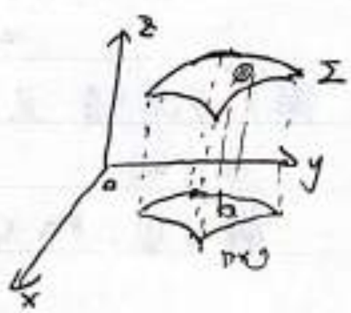
解: Σ: $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$

$(x \geq 0, y \geq 0, z \geq 0)$.

$I = 4\iint_\Sigma (\frac{x}{2} + \frac{y}{3} + \frac{z}{4})\,ds = 4\iint_\Sigma 1\,ds = 4S$

$\vec{AB} = \{-2, 3, 0\}$, $\vec{AC} = \{-2, 0, 4\}$

$\vec{AB} \times \vec{AC} = \{\quad\}$



对 $\iint_\Sigma f(x,y,z)\,ds$

1° Σ: $z = \varphi(x,y)$, $(x,y) \in D_{xy}$;

2° $z'_x = ?$ $z'_y = ?$

$ds = \sqrt{1 + z_x'^2 + z_y'^2}\,d\sigma$

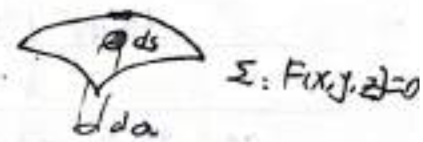
3° $I = \iint_{D_{xy}} f[x, y, \varphi(x,y)] \cdot \sqrt{1 + z_x'^2 + z_y'^2}\,d\sigma$

$\iint f(x,y,z)\,ds$

1°. Σ: $z=\varphi(x,y)$. $(x,y)\in D_{xy}$:

2°. $z_x'=?$ $z_y'=?$

$ds=\sqrt{1+z_x'^2+z_y'^2}\,d\sigma$



3° 原式 $=\iint_{D_{xy}} f[x,y,\varphi(x,y)]\cdot\sqrt{1+z_x'^2+z_y'^2}\,d\sigma$

例2. $I=\iint z\,ds$. Σ: $x^2+y^2+z^2=1$ 被 $z=\sqrt{x^2+y^2}$ 所截.

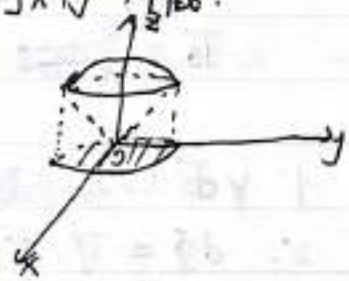
解. 1°: Σ: $z=\sqrt{1-x^2-y^2}$.

$(x,y)\in D_{xy}$: $x^2+y^2\le\frac{1}{2}$

2° $z_x'=-\frac{x}{\sqrt{1-x^2-y^2}}$

$z_y'=-\frac{y}{\sqrt{1-x^2-y^2}}$

$ds=\sqrt{1+z_x'^2+z_y'^2}\,d\sigma=\frac{1}{\sqrt{1-x^2-y^2}}\,d\sigma$

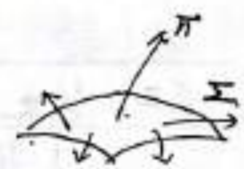


3° $I=\iint_{D_{xy}}\sqrt{1-x^2-y^2}\cdot\frac{1}{\sqrt{1-x^2-y^2}}\,d\sigma=\iint_{D_{xy}}d\sigma=\frac{\pi}{2}$

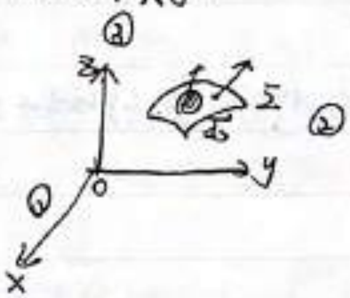
二. 对坐标的曲面积分.

(一) 背景: 流量

$\vec{v}=\{P,Q,R\}$. Σ — 有侧曲面.

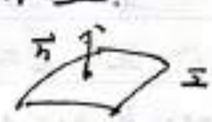


重?



曲线 — 切向量

曲面 — 法向量.



$\forall\,\vec{ds}\subset\Sigma$

1° $\vec{ds}$ 从 x 轴正向往 yOz 面作垂直投影

投影为 $dy\,dz$. 　　　　用切面法.

$$dy\,dz\begin{cases}>0, & \cos\alpha>0\\ <0, & \cos\alpha<0\end{cases}$$

Infact. $dy\,dz=ds\cdot\cos\alpha$;

2°. $\vec{ds}$ 从 y 轴正向往 xoz 平面投影. 投影 dzdx

$$dzdx\begin{cases}>0, & \cos\beta>0\\ <0, & \cos\beta<0\end{cases} \qquad dzdx = ds\cdot\cos\beta;$$

3° $\vec{ds}$ 从 z 轴正向往 xoy 平面投影. 投影 dxdy

$$dxdy\begin{cases}>0, & \cos\gamma>0\\ <0, & \cos\gamma<0\end{cases} \qquad dxdy = ds\cdot\cos\gamma.$$

$\therefore \vec{ds} = \cancel{\{dy,d} \{dydz,\ dzdx,\ dxdy\}$

1° $\forall\, \vec{ds} \subset \Sigma$, $\vec{ds} = \{dydz, dzdx, dxdy\}$;

2°. $d\Phi = \vec{V}\cdot\vec{ds} = Pdydz + Qdzdx + Rdxdy$;

3° $\Phi = \iint\limits_{\Sigma} Pdydz+Qdzdx+Rdxdy$.

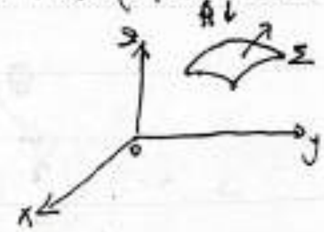
(二) def — $\iint\limits_{\Sigma} Pdydz+Qdzdx+Rdxdy$.

$\iint\limits_{\Sigma} Pdydz$ — $P(x,y,z)$ 在有侧曲面Σ上对坐标y,z的曲面积分.

(三) 性质:

1. $\iint\limits_{\Sigma^-} = -\iint\limits_{\Sigma}$
2. $\iint\limits_{\Sigma} Pdydz+Qdzdx+Rdxdy = \iint\limits_{\Sigma}(P\cos\alpha+Q\cos\beta+R\cos\gamma)ds$

(四) 计算方法.



方法一: 二重积分法.

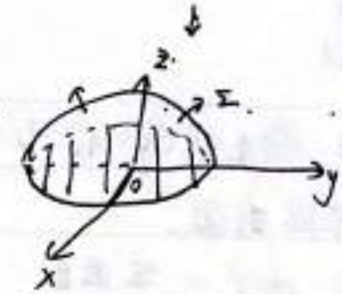
$\iint\limits_{\Sigma} R(x,y,z)\underline{dxdy}$.

1° $\Sigma: z=\varphi(x,y),\ (x,y)\in D_{xy}$;

2° $\iint\limits_{\Sigma} R(x,y,z)\underbrace{\overset{投影}{dxdy}} = \pm\iint\limits_{D_{xy}} R[x,y,\varphi(x,y)]\overset{面积元素}{\underline{dxdy}}$.

$\cos\gamma>0$ 取"+" 或 Σ取上 "+"

$\cos\gamma<0$ 取"−" 或 Σ取下 "−"



例1. $\iint_{\Sigma} z\,dx\,dy$.

解: 1° Σ: $z=\sqrt{4-x^2-y^2}$

D_{xy}: $x^2+y^2\le 4$;

2° $\iint_{\Sigma} z\,dxdy = \iint_{D_{xy}} \sqrt{4-x^2-y^2}\,dxdy = \int_0^{2\pi} d\theta \int_0^2 r\sqrt{4-r^2}\,dr$

$= -\pi \int_0^2 (4-r^2)^{\frac{1}{2}}\,d(4-r^2) = -\pi \times \frac{2}{3}(4-r^2)^{\frac{3}{2}}\Big|_0^2 = -\frac{2}{3}\pi(0-8)$

$= \frac{16\pi}{3}$

方法二:

1-dim: $\int_a^b f(x)dx = F(b)-F(a)$;

2-dim: $\oint_L Pdx+Qdy = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma$

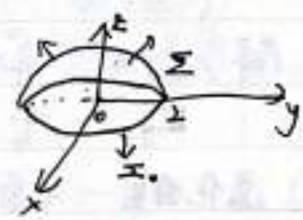
3-dim:

Th(Gauss). 若 ① Ω 一几何体, Σ为Ω的外表面.

② P、Q、R 在Ω上连续可偏导

则 $\oiint_{\Sigma} P\,dydz + Q\,dzdx + R\,dxdy = \iiint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dv$.

例2 $I=\iint_{\Sigma} x^3\,dydz + y^3\,dzdx + 2\,dxdy$.



解: 1° $P=x^3$ $Q=y^3$ $R=2$.

$\frac{\partial P}{\partial x}=3x^2$ $\frac{\partial Q}{\partial y}=3y^2$ $\frac{\partial R}{\partial z}=0$.

2° 补 Σ_0: $z=0$ ($x^2+y^2\le 4$) 下.

$I = \oiint_{\Sigma+\Sigma_0} - \iint_{\Sigma_0}$.

3° $\oiint_{\Sigma+\Sigma_0} = 3\iiint_{\Omega}(x^2+y^2)\,dv = 3\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^2 r^2\sin\varphi\, r^2\sin^2\varphi\,dr$

$= 6\pi \int_0^{\frac{\pi}{2}} \sin^3\varphi\,d\varphi \int_0^2 r^4\,dr = 6\pi \times \frac{2}{3} \times 1 \times \frac{32}{5} = ?$

$\iint_{\Sigma_0} x^3\,dydz + y^3\,dzdx + 2\,dxdy$

$= \iint_{\Sigma_0} 2\,dxdy = -\iint_{D_{xy}} 2\,dxdy = -2\times 4\pi = -8\pi$

直播,

多元微分学的几何应用.

一、空间曲面.

(一) def — Σ曲面，$F(x,y,z)=0$ 方程.

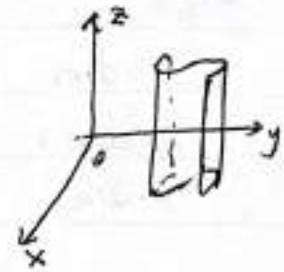
If ① $\forall M(x_0,y_0,z_0)\in\Sigma \Rightarrow F(x_0,y_0,z_0)=0$;

② If (x_0,y_0,z_0) 为 $F(x,y,z)=0$ 任一解 $\Rightarrow (x_0,y_0,z_0)\in\Sigma$

$F(x,y,z)=0$ — Σ的曲面方程，Σ称为 $F(x,y,z)=0$ 图形

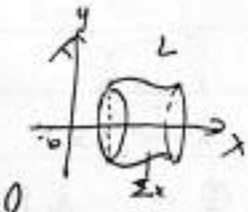
(二) 特殊曲面.

1. 柱面. $\Sigma: F(x,y)=0$ — 母线平行于z轴曲面



2. 2-dim 旋转面.

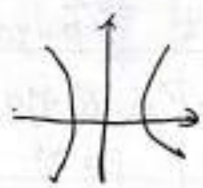
$$L:\begin{cases} F(x,y)=0 \\ z=0 \end{cases}$$



① $\Sigma_x: F(x, \pm\sqrt{y^2+z^2})=0$

② $\Sigma_y: F(\pm\sqrt{x^2+z^2}, y)=0$

例1. $L:\begin{cases} \frac{x^2}{4}-y^2=1 \\ z=0 \end{cases}$ 求 Σ_x, Σ_y.

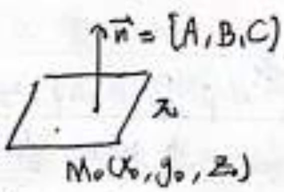


解：Σ_x 为：$\frac{x^2}{4}-y^2-z^2=1$

Σ_y : $\frac{x^2}{4}-y^2+\frac{z^2}{4}=1$

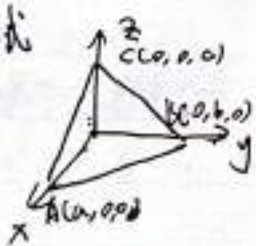
(三) 退化曲面 — 平面.

1. 点法式



$\pi: A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

2. 截距式



$\pi: \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

3. 一般式 $\pi: Ax+By+Cz+D=0$

(四) 空间曲面 $\begin{cases}\text{切平面} \\ \text{法线}.\end{cases}$

$\Sigma: F(x,y,z)=0$

$M_0(x_0,y_0,z_0)\in\Sigma$

$\vec{n}=\{F_x',F_y',F_z'\}_{M_0}$

二、空间曲线.

(一) 形式.

1、一般式　　$L: \begin{cases}F(x,y,z)=0\\ G(x,y,z)=0\end{cases}$

2、参数式

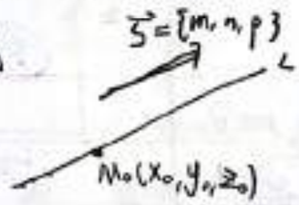
$L: \begin{cases}x=\varphi(t)\\ y=\psi(t)\\ z=\omega(t)\end{cases}$

如: $\begin{cases}x^2+y^2=4\\ x-y-z=1\end{cases}$

$L: \begin{cases}x=2\cos t\\ y=2\sin t\\ z=2\cos t-2\sin t-1\end{cases}$

(二) 退化—直线

1、点向式



$M_0(x_0,y_0,z_0)\in L$

$\vec{s}=\{m,n,p\}\parallel L$

$L: \frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}$

2、参数式.

$L: \begin{cases}x=x_0+mt\\ y=y_0+nt\\ z=z_0+pt\end{cases}$

3、一般式.

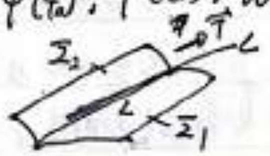
$L: \begin{cases}A_1x+B_1y+C_1z+D_1=0\\ A_2x+B_2y+C_2z+D_2=0\end{cases}$

(三) 空间曲线 $\begin{cases} \text{切线.} \\ \text{法平面.} \end{cases}$

1. L $\begin{cases} x=\varphi(t) \\ y=\psi(t) \\ z=w(t) \end{cases}$ $t=t_0 \Rightarrow M_0(x_0,y_0,z_0)\in L.$

$$\vec{T}=\{\varphi'(t_0),\ \psi'(t_0),\ w'(t_0)\}.$$

2. L $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$



$M_0(x_0,y_0,z_0)\in L$

$\vec{n_1}=\{F'_x,\ F'_y,\ F'_z\}_{M_0}$

$\vec{n_2}=\{G'_x,\ G'_y,\ G'_z\}_{M_0}.$

$\vec{T}=\vec{n_1}\times\vec{n_2}\Big|_{M_0}$

方向导数与梯度.

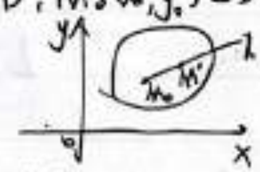
一、背景: $\Sigma: z=\varphi(x,y)$. $(x,y)\in D$. $M_0(x_0,y_0)\in D$.

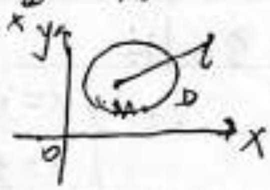
二、方向导数定义.

1. (二元) — $z=f(x,y)$, $(x,y)\in D$. $M_0(x_0,y_0)\in D$.

过 M_0 作射线 l

$M'(x+\Delta x,\ y_0+\Delta y)\in l$





$\Delta z=f(x_0+\Delta x,\ y_0+\Delta x)-f(x_0,y_0)$

$\rho=|M_0M'|=\sqrt{(\Delta x)^2+(\Delta y)^2}$

If $\lim\limits_{\rho\to0}\frac{\Delta z}{\rho}$ ∃ 称此极限为 $z=f(x,y)$ 在 M_0 沿射线 l 的方向导数.

记 $\frac{\partial z}{\partial l}\Big|_{M_0}$ 即 $\frac{\partial z}{\partial l}\Big|_{M_0}\triangleq\lim\limits_{\rho\to0}\frac{\Delta z}{\rho}$.

2. (三元) — $u=f(x,y,z)$.

$(x,y,z)\in\Omega$. $M_0(x_0,y_0,z_0)\in\Omega$

过 M_0 作射线 l

$M'(x_0+\Delta x,\ y_0+\Delta y,\ z+\Delta z)\in l$

$|M_0M'|=\rho=\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}$

$\Delta u=f(M')-f(M_0)$. If $\lim\limits_{\rho\to0}\frac{\Delta u}{\rho}$ ∃. 称此极限为 $u=f(x,y,z)$ 在 M_0 处沿 l 的方向导数 记 $\frac{\partial u}{\partial l}\Big|_{M_0}$

$\frac{\partial u}{\partial l}\Big|_{M_0} = \lim_{\rho\to 0}\frac{\Delta u}{\rho}$

例1. $z = x^2\sin y$ 在 $(2,\frac{\pi}{2})$ 处沿从 $A(2,\frac{\pi}{2})$ 到 $B(3,\pi)$ 方向的方向导数

解: 1° $\frac{\partial z}{\partial x} = 2x\sin y$ $\quad \frac{\partial z}{\partial y} = x^2\cos y$ $\quad$ 2° $\overrightarrow{AB} = \{1,\frac{\pi}{2}\}$

$\frac{\partial z}{\partial x}\Big|_A = 4.$ $\quad \frac{\partial z}{\partial y}\Big|_A = 0$ $\quad \cos\alpha = \frac{1}{\sqrt{1+\frac{\pi^2}{4}}}$ $\quad \cos\beta = \frac{\frac{\pi}{2}}{\sqrt{1+\frac{\pi^2}{4}}}$

3° $\frac{\partial z}{\partial l}\Big|_A = 4\times\frac{2}{\sqrt{\pi^2+4}} + 0\times\cdots$

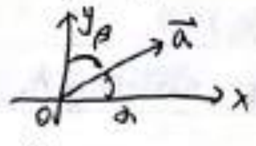
Note:

① $\vec{a} = \{a_1, b_1, c_1\}$ $\quad |\vec{a}| \triangleq \sqrt{a_1^2+b_1^2+c_1^2}$

If $|\vec{a}| = 1$, $\vec{a}$ — 单位向量

If 若 $\vec{a} \neq 0$ $\quad \underline{\vec{a}^{\circ} \triangleq \frac{1}{|\vec{a}|}\vec{a}}$ $\quad \vec{a}^{\circ} = \{\frac{a_1}{|\vec{a}|}, \frac{b_1}{|\vec{a}|}, \frac{c_1}{|\vec{a}|}\}$

②

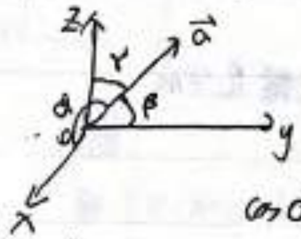


α, β — 方向角. $\quad \cos\alpha, \cos\beta$ — 方向余弦

$\vec{a} = \{a_1, b_1\}$. $\quad \cos\alpha = \frac{a_1}{|\vec{a}|}$. $\quad \cos\beta = \frac{b_1}{|\vec{a}|}$

$\{\cos\alpha, \cos\beta\} = \vec{a}^{\circ}$

③



α, β, γ — 方向角. $\quad \cos\alpha, \cos\beta, \cos\gamma$ — 方向余弦

设 $\vec{a} = \{a_1, b_1, c_1\}$

$\cos\alpha = \frac{a_1}{|\vec{a}|}$, $\quad \cos\beta = \frac{b_1}{|\vec{a}|}$, $\quad \cos\gamma = \frac{c_1}{|\vec{a}|}$

$\{\cos\alpha, \cos\beta, \cos\gamma\} = \vec{a}^{\circ}$

三、方向导数计算

1. $z = f(x,y)$. $\quad (x,y)\in D$. $M_0(x_0,y_0)\in D$

过 M_0 作射线 l

$M'(x_0+\Delta x, y_0+\Delta y)\in l$

$\frac{\partial z}{\partial l}\Big|_{M_0} \triangleq \lim_{\rho\to 0}\frac{\Delta z}{\rho}$

$= \frac{\partial z}{\partial x}\Big|_{M_0}\cos\alpha + \frac{\partial z}{\partial y}\Big|_{M_0}\cos\beta.$

四、梯度

$\frac{\partial u}{\partial l}\Big|_{M_0} = \frac{\partial u}{\partial x}\Big|_{M_0}\cos\alpha + \frac{\partial u}{\partial y}\Big|_{M_0}\cos\beta + \frac{\partial u}{\partial z}\Big|_{M_0}\cos\gamma$

$= \underbrace{\{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\}_{M_0}}_{\text{固定向量}} \cdot \underbrace{\{\cos\alpha, \cos\beta, \cos\gamma\}}_{\text{单位向量 方向与}l\text{相同}} = \vec{e}$

$\frac{\partial u}{\partial l}\Big|_{M_0} = \left|\{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\}_{M_0}\right| \cdot \underbrace{|\vec{e}|}_{1} \cdot \cos\theta$

$\vec{a}=\{a_1, b_1, c_1\}$ $\vec{b}=\{a_2, b_2, c_2\}$

$\vec{a}\cdot\vec{b} = a_1a_2+b_1b_2+c_1c_2$

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\langle\widehat{\vec{a},\vec{b}}\rangle$

$\cdots\ \frac{\partial u}{\partial l}\Big|_{M_0} = \underbrace{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2}\Big|_{M_0}}_{\text{constant}}\cdot\underline{\cos\theta}$

$\theta=0$ 时，$\frac{\partial u}{\partial l}\Big|_{M_0}$ 取最大值.

$\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2}\Big|_{M_0}$

$\left\{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right\}\Big|_{M_0} \triangleq \text{grad}\, u\big|_{M_0}$

$u=f(x,y,z)$ 在 M_0 处的梯度.

若 l 的方向与梯度方向同，方向导数最大，函数增长最快.

定积分的物理应用 $\begin{cases}\text{力 — 压力. 引力}\\ \text{功}\end{cases}$

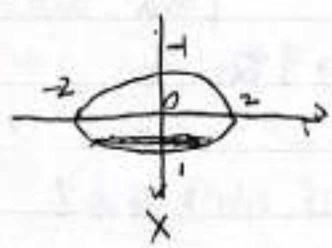
一、力.

长轴长 4m

1. 一洒水车底为椭圆.（高 2m，水平 4m）

(1) 装满水，水对底压力？

(2) 全部抽干，做功几何？



解(1) 1° 取 $[x, x+dx] \subset [-1, 1]$

2° $dF = \rho g(x+1)(y_2-y_1)dx$

由 $x^2+\frac{y^2}{4}=1 \Rightarrow y=\pm 2\sqrt{1-x^2}$

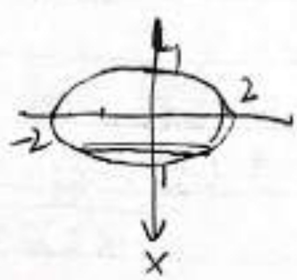
$dF = \rho g(x+1)\cdot 4\sqrt{1-x^2}\,dx$

3° $F = 4\rho g\int_{-1}^{1}(x+1)\sqrt{1-x^2}\,dx$

$= 4\rho g\int_{0}^{1}\sqrt{1-x^2}\,dx$

$= 8\rho g\times\frac{\pi}{4} = 2\pi\rho g$

(2)



1° 取 $[x, x+dx] \subset [-1, 1]$；

2° $dw = (x+1)\cdot\rho g\,dv$

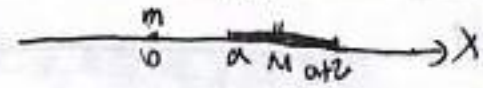
$= (x+1)\rho g(y_2-y_1)\times dx\times 4$

$=16\rho g(x+1)\sqrt{1-x^2}\,dx$

3° $W=16\rho g\int_{-1}^{1}(x+1)\sqrt{1-x^2}\,dx$

$=32\rho g\int_{0}^{1}\sqrt{1-x^2}\,dx$

$=8\pi\rho g$

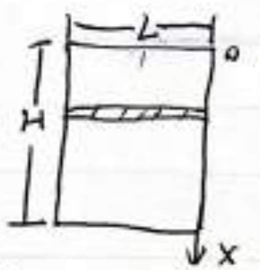
2. 

求位于原点质量为m质点与细棒引力大小？

解：1° 取 $[x, x+dx]\subset[a, a+L]$

2° $dF = k\dfrac{m\times\frac{M}{L}\times dx}{x^2}$

3° $F=\int_a^{a+L}dF = \dfrac{kmM}{L}\int_a^{a+L}\dfrac{dx}{x^2} = \dfrac{kmM}{L}\left(\dfrac{1}{a}-\dfrac{1}{a+L}\right)$

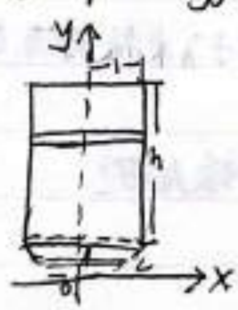
3. 

解：如图

1° 取 $[x, x+dx]\subset[0, H]$

2° $dF=\rho g\times x\times Ldx$

3° $F=\int_0^H dF=\rho gL\int_0^H x\,dx=\frac{1}{2}\rho gLH^2$

4. 

$F_上 : F_下 = 5:4$. 求h

解① $F_上$：

1° 取 $[y, y+dy]\subset[1, h+1]$

2° $dF_上=\rho g(h+1-y)\cdot 2\cdot dy$

$=2\rho g(h+1-y)dy$;

3° $F_上=2\rho g\int_1^{h+1}(h+1-y)\,dy=2\rho g\left[h(h+1)-\frac{1}{2}(h^2+2h)\right]$

② $L: y=x^2$

1° 取 $[y, y+dy] \subset [0,1]$.

2° $dF_下 = \rho g(h+1-y)\cdot(x_2-x_1)\cdot dy$

$= 2\rho g(h+1-y)\sqrt{y}\,dy$

3° $F_下 = 2\rho g\int_0^1 (h+1-y)\sqrt{y}\,dy$

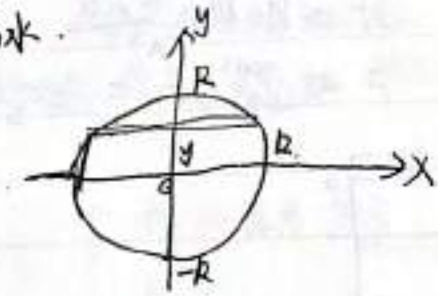
$= 4\rho g(\frac{1}{3}h+\frac{2}{15})$

③ $F_上 : F_下 = 5:4 \Rightarrow h=2$

(二) 做功.

1. 半径为R的球体盛满水.
抽干水做功几何?

解:



1° 取 $[y, y+dy] \subset [-R, R]$

2° $dw = \rho g dv(R-y) = \rho g(R-y)\cdot \pi x^2 dy = \pi\rho g(R-y)\cdot(R^2-y^2)dy$

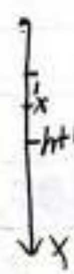
3° $w = \pi\rho g\int_{-R}^{R}(R-y)(R^2-y^2)dy$

$= 2\pi R\rho g\int_0^R (R^2-y^2)dy$

$= \frac{4}{3}\pi R^4\rho g$

2、往木板钉钉子. 木板对钉子阻力与钉子入木板深度成正比.
(比例系数k) 每锤做功相同.
已知第一锤第一次入木板 1cm 问第二次入木板几何?

解: 设第二次入木板 h cm.



① W_1:

1° 取 $[x, x+dx] \subset [0,1]$;

2° $dw_1 = kx\,dx$

3° $W_1 = k\int_0^1 x\,dx = \frac{k}{2}$;

② W_2

1°. 取 $[x, x+dx] \subset [1, h+1]$.

2° $dw_2 = kx\,dx$

3° $w_2 = k\int_1^{h+1} x\,dx = \frac{k}{2}[(h+1)^2-1]$.

由 $\frac{k}{2}[(h+1)^2-1] = \frac{k}{2} \Rightarrow h=\sqrt{2}-1$ (cm)

~~多元函数的几何应用~~

~~一、空间曲面~~

~~(一) def~~

第一模块 极限

理论部分

Part I 极限

一、defs.

极限 $\begin{cases} \varepsilon-N \\ \varepsilon-\delta \\ \varepsilon-X \end{cases}$

例1. 证明：$\lim\limits_{x\to 1}\frac{(2x^2-x-1)}{x-1}=3$

证：$\forall \varepsilon>0$

$\left|\frac{2x^2-x-1}{x-1}-3\right|=|2(x-1)|=2|x-1|<\varepsilon$

$\Leftrightarrow 0<|x-1|<\frac{\varepsilon}{2}$

$\exists \delta=\frac{\varepsilon}{2}>0$，当 $0<|x-1|<\delta$ 时

$\left|\frac{2x^2-x-1}{x-1}-3\right|<\varepsilon$

$\therefore \lim\limits_{x\to 1}\frac{2x^2-x-1}{x-1}=3$

如：$\lim\limits_{n\to\infty}a_n=a\ (\neq 0)$

当 n 充分大时，$|a_n|<\frac{a}{2}$． $|a_n|>\frac{a}{2}$ …… ？

$\lim\limits_{n\to\infty}a_n=a \ \underset{\not\Leftarrow}{\Rightarrow}\ \lim\limits_{n\to\infty}|a_n|=|a|$

取 $\varepsilon=\frac{|a|}{2}>0$．

$\because \lim\limits_{n\to\infty}|a_n|=|a|$，$\therefore \exists N>0$，当 $n>N$ 时

$\big||a_n|-|a|\big|<\frac{|a|}{2} \Rightarrow \frac{|a|}{2}<|a_n|<\frac{3}{2}|a|$

2、无穷小 — $\lim\limits_{x\to a}\alpha(x)=0$

$\alpha(x)$ 为当 $x\to a$ 时的无穷小

$\alpha\to 0$，$\beta\to 0$

If $\lim\frac{\beta}{\alpha}=0$ $\quad \beta=o(\alpha)$

If $\lim\frac{\beta}{\alpha}=k\ (\neq 0,\infty)$，$\beta=O(\alpha)$

If $\lim\frac{\beta}{\alpha}=1$ $\quad \alpha\sim\beta$．

If $\lim\limits_{x\to 0}\frac{\alpha}{x^k}=c\ (\neq 0,\infty)$，$\alpha$ 称为 x 的 k 阶无穷小．

注：x，$\sin x$，$\tan x$，$\arcsin x$，$\arctan x$

任两个之差为3阶无穷小.

二. 性质

(一) 一般：

1. (唯一性)

2. (保号性) $\lim\limits_{x\to a} f(x)=A \begin{cases}>0\\<0\end{cases}$ 则

$\exists\,\delta>0$，当 $0<|x-a|<\delta$ 时，$f(x)\begin{cases}>0\\<0\end{cases}$

例2. $f'(0)=0$，$\lim\limits_{x\to 0}\dfrac{f''(x)}{x^2+x^3}=-1$，$x=0$?

解：$\exists\,\delta>0$，当 $0<|x|<\delta$ 时

$$\frac{f''(x)}{x^2+x^3}<0 \Rightarrow f''(x)<0$$

$\Rightarrow f'(x)$ 在 $(-\delta,\delta)\downarrow$.

$\because f'(0)=0$

$\therefore \begin{cases} f'(x)>0, & -\delta<x<0 \\ f'(x)<0, & 0<x<\delta \end{cases} \Rightarrow x=0$ 为极大点

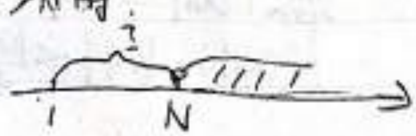
3. (有界)

① $\lim\limits_{n\to\infty} a_n=A \underset{\not\Leftarrow}{\Rightarrow} \exists\, M>0,\ |a_n|\le M$

证明：取 $\varepsilon=1$，$\exists\, N>0$，当 $n>N$ 时

$|a_n-A|<1$

$\Rightarrow |a_n|<1+|A|$



取 $M=\max\{|a_1|, \cdots |a_N|, 1+|A|\}$

$\forall n$，$|a_n|\le M$

"$\not\Leftarrow$" 如 $a_n=(-1)^n$

$\lim\limits_{n\to\infty} a_n$ 无　　$|a_n|=1$

③(局部有界). If $\lim\limits_{x\to a} f(x)=A$，则

$\exists\,\delta>0$, $M>0$，当 $0<|x-a|<\delta$ 时

$|f(x)|\le M$

证：取 $\varepsilon=1$.

$\because \lim\limits_{x\to a} f(x)=A$ $\quad\therefore \exists\,\delta>0$. 当 $|x-a|<\delta$ 时.

$|f(x)-A|<1$

$\Rightarrow |f(x)|<1+|A| \triangleq M$

4. ① 列$\exists$极限 $\Rightarrow$ 任一子列$\exists$同样极限

② 子列$\exists$极限 $\not\Rightarrow$ 列$\exists$极限

例3. $\lim\limits_{n\to+\infty}[n-n^2\ln(1+\frac{1}{n})]$

| $x\to+\infty$ 列
| $n\to+\infty$ 子列

解：$\lim\limits_{x\to+\infty}[x-x^2\ln(1+\frac{1}{x})]$

$=\lim\limits_{x\to+\infty}\dfrac{\frac{1}{x}-\ln(1+\frac{1}{x})}{\frac{1}{x^2}}=\lim\limits_{t\to0}\dfrac{t-\ln(1+t)}{t^2}$

$\because \ln(1+t)=t-\frac{t^2}{2}+o(t^2)$

$\therefore t-\ln(1+t)\sim\frac{1}{2}t^2$

$\therefore$ 原式 $=\frac{1}{2}$

另法：$\ln(1+\frac{1}{x})=\frac{1}{x}-\frac{1}{2x^2}+o(\frac{1}{x^2})$

$x-x^2\ln(1+\frac{1}{x})=\frac{1}{2}+\dfrac{o(\frac{1}{x^2})}{\frac{1}{x^2}}$

$\therefore$ 原式 $=\frac{1}{2}$

例4. $\lim\limits_{x\to0}\frac{1}{x}\cos\frac{1}{x}$ (　　　)

(A) 0　　(B) 1　　(C) ∞　　(D) 不存在 但不是 ∞ ✓

解：取 $x_n=\frac{1}{2n\pi}\to0\ (n\to\infty)$

| $x\to0$ 列

$\lim\limits_{n\to\infty}\frac{1}{x_n}\cos\frac{1}{x_n}=\lim\limits_{n\to\infty}2n\pi\cos2n\pi=+\infty$

取 $y_n=\frac{1}{2n\pi+\frac{\pi}{2}}\to0\ (n\to\infty)$

$\lim\limits_{n\to\infty}\frac{1}{y_n}\cos\frac{1}{y_n}=\lim\limits_{n\to\infty}(2n\pi+\frac{\pi}{2})\underbrace{\cos(2n\pi+\frac{\pi}{2})}_{0}=0$

Note: ① $\lim\limits_{n\to\infty}a_n=A \;\overset{\Rightarrow}{\not\Leftarrow}\; \lim\limits_{n\to\infty}|a_n|=|A|$

② $\lim\limits_{n\to\infty}a_n=A \Leftrightarrow \lim\limits_{n\to\infty}a_{2n}=A,\ \lim\limits_{n\to\infty}a_{2n+1}=A$

(三) ∃性质

1. 迫敛定理

① 数列型

② 函数型：If $\begin{cases} f(x) \le g(x) \le h(x) \\ \lim f(x) = \lim h(x) = A \end{cases} \Rightarrow \lim g(x) = A$

Note：$\begin{cases} f(x) \le g(x) \le h(x) \\ \lim [h(x) - f(x)] = 0 \end{cases}$

$\overset{?}{\Rightarrow} \lim g(x)$ ∃ ×

反例：$\begin{cases} e^x - e^{-x} \le e^x \le e^x + e^{-x} \\ \lim\limits_{x\to+\infty} (h - f) = \lim\limits_{x\to+\infty} 2e^{-x} = 0 \end{cases}$

但 $\lim\limits_{x\to+\infty} g(x) = +\infty$

2. 单调有界数列∃极限

Case 1. $\{a_n\}\uparrow$

$\begin{cases} \{a_n\} \text{无上界} \Rightarrow \lim\limits_{n\to\infty} a_n = +\infty \\ \text{若 } a_n \le M \Rightarrow \lim\limits_{n\to\infty} a_n \ \exists. \end{cases}$

Case 2. $\{a_n\}\downarrow$

$\begin{cases} \{a_n\} \text{无下界} \Rightarrow \lim\limits_{n\to\infty} a_n = -\infty \\ a_n \ge M \Rightarrow \lim\limits_{n\to\infty} a_n \ \exists \end{cases}$

Note：设 $a_{n+1} = f(a_n)$.

令 $y = f(x)$

If $f'(x) > 0 \Rightarrow \{a_n\}$ 单调 $\begin{cases} a_1 < a_2, \uparrow \\ a_1 > a_2, \downarrow \end{cases}$

证：$f'(x) > 0 \Rightarrow f(x)\uparrow$

case 1. $a_1 < a_2$

$\Rightarrow f(a_1) < f(a_2)$ 即 $a_2 < a_3$

设 $a_k < a_{k+1}$

$\Rightarrow f(a_k) < f(a_{k+1})$ 即 $a_{k+1} < a_{k+2}$

$\therefore \forall n,\ a_n < a_{n+1}$. 即 $\{a_n\}\uparrow$.

Case 2. $a_1 > a_2$

$\Rightarrow f(a_1) > f(a_2)$，即 $a_2 > a_3$.

设 $a_k > a_{k+1}$.

$\Rightarrow f(a_k) > f(a_{k+1})$，即 $a_{k+1} > a_{k+2}$

$\therefore \forall n \quad a_n > a_{n+1}$ 即 $\{a_n\}\downarrow$.

例5. $a_1 = 2 \quad a_{n+1} = \frac{a_n+1}{a_n+2}$ 证 $\lim\limits_{n\to\infty} a_n \exists$.

证：令 $y = \frac{x+1}{x+2}$

$y' = \frac{x+2-(x+1)}{(x+2)^2} = \frac{1}{(x+2)^2} > 0$

$a_1 = 2 > a_2 = \frac{3}{4} \Rightarrow \{a_n\}\downarrow$

又 $a_n > 0$

$\therefore \lim\limits_{n\to\infty} a_n \exists$

(三) 无穷小性质：

1. 一般：

① $\alpha \to 0,\ \beta \to 0 \Rightarrow \begin{cases} \alpha \pm \beta \to 0 \\ \alpha\beta \to 0 \\ k\alpha \to 0 \end{cases}$

② $|\alpha| \le M,\ \beta \to 0 \Rightarrow \alpha\beta \to 0$.

③ $\lim f(x) = A \Leftrightarrow f(x) = A + \alpha,\ \alpha \to 0$

2. 等价性质：

① $\begin{cases} \alpha \sim \alpha_0,\ \beta \sim \beta_0 \\ \lim \frac{\beta_0}{\alpha_0} = A \end{cases} \Rightarrow \lim \frac{\beta}{\alpha} = A$

② $\alpha \sim \beta \Leftrightarrow \beta = \alpha + o(\alpha)$

证："$\Rightarrow$" $\alpha \sim \beta \Rightarrow \lim \frac{\beta}{\alpha} = 1 \Rightarrow \frac{\beta}{\alpha} = 1 + r,\ r \to 0$

$\Rightarrow \beta = \alpha + \alpha r$

$\because \lim \frac{\alpha r}{\alpha} = 0 \quad \therefore \alpha r = o(\alpha)$

"$\Leftarrow$" $\beta = \alpha + o(\alpha) \Rightarrow \frac{\beta}{\alpha} = 1 + \frac{o(\alpha)}{\alpha}$

$\lim \frac{\beta}{\alpha} = 1 \Rightarrow \alpha \sim \beta$

3. $x \to 0$：

① $x \sim \sin x \sim \tan x \sim \arcsin x \sim \arctan x \sim e^x - 1 \sim \ln(1+x)$；

② $\begin{cases} 1-\cos x \sim \frac{1}{2}x^2 \\ 1-\cos^a x \sim \frac{a}{2}x^2 \end{cases}$；

③ $(1+\Delta)^a-1 \sim a\Delta \quad (\Delta \to 0)$；

④ $a^x-1 \sim x\ln a$

$(a^x-1 = e^{x\ln a}-1 \sim x\ln a)$.

4. $x \to 0$:

① $e^x = 1+x+\frac{x^2}{2!}+\cdots$

② $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

③ $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$

④ $\frac{1}{1-x} = 1+x+x^2+\cdots$

⑤ $\frac{1}{1+x} = 1-x+x^2-\cdots$

⑥ $\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$

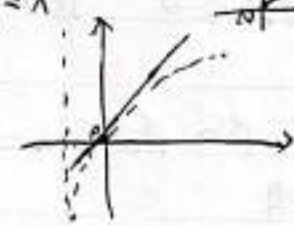
⑦ $\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$

五. 重要极限

Notes:

① $\begin{cases} 0<x<\frac{\pi}{2}\text{时} & \sin x<x<\tan x. \\ x\geq 0\text{时} & \sin x \leq x \end{cases}$

② $\begin{cases} x>0\text{时}. & \ln(1+x)<x \\ x>-1\text{时}. & \ln(1+x)\leq x \end{cases}$



(Ⅰ) $\lim\limits_{\Delta\to 0}\frac{\sin\Delta}{\Delta}=1$

(Ⅱ) $\lim\limits_{\Delta\to 0}(1+\Delta)^{\frac{1}{\Delta}}=e$

Part Ⅱ 连续与间断

一. defs:

1. 连续:

① $\lim\limits_{x\to a}f(x)=f(a) \Leftrightarrow f(a-0)=f(a+0)=f(a)$

② $\begin{cases} f(x)\text{在}(a,b)\text{内处处连续} \\ f(a)=f(a+0),\ f(b)=f(b-0) \end{cases}$

称 $f(x)\in C[a,b]$

2. 间断 — If $\lim\limits_{x\to a} f(x) \neq f(a)$

分类:

第一类: $f(a-0)$; $f(a+0)$ $\exists$

$$\begin{cases} f(a-0)=f(a+0)\ (\neq f(a)) & \text{— } a \text{ 可去间断点} \\ f(a-0)\neq f(a+0) & \text{— } a \text{ 跳跃间断点.} \end{cases}$$

第二类: $f(a-0)$, $f(a+0)$ 至少一个不存在.

二. $f(x)\in C[a,b]$:

1. $\exists\, m, M$.
2. $|f(x)|\le k$.
3. If $f(a)f(b)<0$, $\exists\, c\in(a,b)$, $f(c)=0$.
4. $\forall \eta\in[m,M]$, $\exists\, \xi\in[a,b]$, $f(\xi)=\eta$.

题一 n项和、积极限

① 先和、积再极限

1. $\lim\limits_{n\to\infty}\left[\sum\limits_{k=1}^{n}\frac{2}{(2k-1)(2k+1)}\right]^n$

解: $\sum\limits_{k=1}^{n}\frac{2}{(2k-1)(2k+1)}=(1-\frac{1}{3})+(\frac{1}{3}-\frac{1}{5})+\cdots+(\frac{1}{2n-1}-\frac{1}{2n+1})=1-\frac{1}{2n+1}$

原式 $=\lim\limits_{n\to\infty}\left[\left(1-\frac{1}{2n+1}\right)^{-(2n+1)}\right]^{-\frac{n}{2n+1}}=e^{-\frac{1}{2}}$

② 夹…

③ 定积分定义 $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}f(\frac{i}{n})=\int_0^1 f(x)\,dx$

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}f\left(a+\frac{(b-a)i}{n}\right)=\int_a^b f(x)\,dx.$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{kn}f\left(\frac{i}{n}\right)=\int_0^k f(x)\,dx$$

例2: $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{1}{i+\sqrt{n^2-i^2}}$

解. 原式 $=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{\frac{i}{n}+\sqrt{1-(\frac{i}{n})^2}}=\int_0^1\frac{dx}{x+\sqrt{1-x^2}}$

$\xlongequal{x=\sin t}\int_0^{\frac{\pi}{2}}\frac{\cos t}{\sin t+\cos t}\,dt=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}\,dx\triangleq I$

$=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\cos x+\sin x}\,dx$

$2I=\frac{\pi}{2}\implies I=\frac{\pi}{4}$

例3 $\lim\limits_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+2n}\right)$

$=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{2n}\frac{1}{n+i}=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{2n}\frac{1}{1+\frac{i}{n}}=\int_0^2\frac{dx}{1+x}=\ln 3.$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

例4. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{i}{n^2}\sin^4\frac{\pi i}{n} = \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\frac{i}{n}\sin^4\frac{\pi i}{n}$

$= \int_0^1 x\sin^4\pi x\,dx = \frac{1}{\pi^2}\int_0^{\pi} x\sin^4 x\,dx$

$= \frac{1}{\pi^2}\times\frac{\pi}{2}\int_0^{\pi}\sin^4 x\,dx = \frac{1}{2\pi}\times 2\times I_4 = \frac{1}{\pi}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{3}{16}$

$\frac{1}{\pi^2}\int_0^1 \pi x\sin^4\pi x\,d(\pi x)$

$\frac{1}{\pi^2}\int_0^{\pi} x\sin^4 x\,dx$

$\frac{1}{\pi^2}\times\frac{\pi}{2}\int_0^{\pi}\sin^4 x\,dx$

例5. $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim\limits_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(\frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n}\right)^{\frac{1}{n}}$

$= e^{\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\ln\frac{i}{n}} = e^{\int_0^1 \ln x\,dx}$

而 $\int_0^1 \ln x\,dx = x\ln x\Big|_0^1 - \int_0^1 x\cdot\frac{1}{x}\,dx$

$\because \lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$

$\therefore \int_0^1 \ln x\,dx = -1 \quad \therefore$ 原式 $= e^{-1}$

例6. $\lim\limits_{n\to\infty}\left[\frac{(1+\cos\frac{\pi}{n})^2}{n+1} + \frac{(1+\cos\frac{2\pi}{n})^2}{n+\frac{1}{2}} + \cdots + \frac{(1+\cos\frac{n\pi}{n})^2}{n+\frac{1}{n}}\right]$

解：令 $[\cdots] = b_n$

$\frac{n}{n+1}\times\frac{1}{n}\sum\limits_{i=1}^{n}(1+\cos\frac{\pi i}{n})^2 \le b_n \le \frac{1}{n}\sum\limits_{i=1}^{n}(1+\cos\frac{\pi i}{n})^2$

先夹逼，后定义

$\because \lim\limits_{n\to\infty}\frac{n}{n+1} = 1$

$\therefore \lim\limits_{n\to\infty}$ 左 $= \lim\limits_{n\to\infty}$ 右 $= \int_0^1 (1+\cos\pi x)^2\,dx$

$\therefore$ 原 $= \int_0^1 (1+\cos\pi x)^2\,dx$

$= \frac{1}{\pi}\int_0^{\pi}(1+\cos x)^2\,dx$

$= \frac{1}{\pi}\int_0^{\pi}(2\cos^2\frac{x}{2})^2\,dx$

$= \frac{8}{\pi}\int_0^{\pi}\cos^4\frac{x}{2}\,d(\frac{x}{2})$

$= \frac{8}{\pi}\int_0^{\frac{\pi}{2}}\cos^4 x\,dx$

$= \frac{8}{\pi}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$

$= \frac{3}{2}$

题二：数列极限存在性证明

1. ① 证：$x+x^2+\cdots+x^n=\frac{1}{2}$ ∃1正根 x_n；

② 证：$\lim_{n\to\infty} x_n$ ∃，求出极限.

$n=1$，$x=\frac{1}{2}$	$x_1=\frac{1}{2}$
$n=2$，$x+x^2=\frac{1}{2}$	$x_2>0$
$n=3$，$x+x^2+x^3=\frac{1}{2}$	$x_3>0$
⋮	数列

证：① 令 $f(x)=x+x^2+\cdots+x^n-\frac{1}{2}$

$f(0)=-\frac{1}{2}$　$f(1)=n-\frac{1}{2}>0$

$\because f(0)f(1)<0$

$\therefore \exists\, x_n\in(0,1)$，使 $f(x_n)=0$

$\because f'(x)=1+2x+\cdots+nx^{n-1}>0\ (x>0)$

$\therefore f(x)$ 在 $(0,+\infty)\uparrow$

$\therefore x_n$ 唯一

② $\begin{cases} x_n+x_n^2+\cdots x_n^n=\frac{1}{2} \\ x_{n+1}+x_{n+1}^2+\cdots x_{n+1}^n+x_{n+1}^{n+1}=\frac{1}{2} \end{cases}$

$\Rightarrow (x_{n+1}-x_n)+(x_{n+1}^2-x_n^2)+\cdots+(x_{n+1}^n-x_n^n)=-x_{n+1}^{n+1}<0$

$\Rightarrow x_{n+1}<x_n \Rightarrow \{x_n\}\downarrow$

又 $x_n>0$

$\therefore \lim_{n\to\infty} x_n$ ∃

令 $\lim_{n\to\infty} x_n=a$　$\left(\begin{matrix} 0<x_n\le\frac{1}{2} \\ 0<x_n^n\le\frac{1}{2^n}\end{matrix}\right)$

$\frac{x_n(1-x_n^n)}{1-x_n}=\frac{1}{2}$

$\Rightarrow \frac{a}{1-a}=\frac{1}{2}$

例4. $a_1=2$　$a_{n+1}=\frac{1}{2}(a_n+\frac{1}{a_n})$　证明 $\lim_{n\to\infty} a_n$ 存在

$a_{n+1}\ge 1$

$a_{n+1}-a_n=\frac{1}{2}(a_n+\frac{1}{a_n})-a_n=\frac{1-a_n^2}{2a_n}\le 0$

$\Rightarrow \{a_n\}\downarrow \Rightarrow \lim_{n\to\infty} a_n$ ∃

例5. $0<a_1<2$ 又 $a_{n+1}=\sqrt{a_n(2-a_n)}$　证明 $\lim_{n\to\infty} a_n$ 存在

$a_{n+1}\le\frac{a_n+2-a_n}{2}=1$

$a_{n+1}-a_n=\sqrt{a_n(2-a_n)}-a_n=\frac{2a_n(1-a_n)}{\sqrt{a_n(2-a_n)}+a_n}\ge 0 \Rightarrow \{a_n\}\uparrow$

$\therefore \lim_{n\to\infty} a_n$ ∃.

P_{21} 到7. (1) 证：$x>0$ 时 $\frac{x}{1+x}<\ln(1+x)<x$

证：法一：

$f(x)=x-\ln(1+x)$. $f(0)=0$

$f'(x)=1-\frac{1}{1+x}>0\ (x>0)$

$\begin{cases} f(0)=0 \\ f'(x)>0 \quad (x>0) \end{cases} \Longrightarrow f(x)>0 \quad (x>0)$

$g(x)=\ln(1+x)-\frac{x}{1+x} \quad g(0)=0$

$g'(x)=\frac{1}{1+x}-\frac{1}{(1+x)^2}>0 \quad x>0$

$\begin{cases} g(0)=0 \\ g'(x)>0 \quad (x>0) \end{cases} \Longrightarrow g(x)>0 \quad (x>0)$

法二：令 $\varphi(x)=\ln(1+x) \qquad \varphi'(x)=\frac{1}{1+x}$

$x>0 \quad \varphi(x)=\varphi(x)-\varphi(0)=\varphi'(\xi)x=\frac{x}{1+\xi} \quad (0<\xi<x)$

$\because \frac{x}{1+x}<\frac{x}{1+\xi}<x$

$\therefore \frac{x}{1+x}<\ln(1+x)<x$

(2) 设 $a_n=1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n$ 证明：$\lim\limits_{n\to\infty}a_n\ \exists$

$a_{n+1}-a_n=[1+\cdots\frac{1}{n+1}-\ln(n+1)]-[1+\cdots+\frac{1}{n}-\ln n]$

$=\frac{1}{n+1}-\ln(1+\frac{1}{n})$

$\because x>0$ 时 $\ln(1+x)>\frac{x}{1+x}$

$\therefore \ln(1+\frac{1}{n})>\frac{\frac{1}{n}}{1+\frac{1}{n}}=\frac{1}{n+1}$

$\therefore a_{n+1}-a_n<0 \Rightarrow \{a_n\}\downarrow$

$x\in[1,2]$ 时. $1\geqslant\frac{1}{x} \Longrightarrow 1\geqslant\int_1^2\frac{1}{x}dx$

$x\in[2,3]$ 时. $\frac{1}{2}\geqslant\frac{1}{x} \Longrightarrow \frac{1}{2}\geqslant\int_2^3\frac{1}{x}dx$

$\cdots$

$x\in[n,n+1]$ 时 $\frac{1}{n}\geqslant\frac{1}{x} \Longrightarrow \frac{1}{n}\geqslant\int_n^{n+1}\frac{1}{x}dx$

$\underline{1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n} \geqslant \ln(n+1)-\ln n$

$a_n\geqslant 0$

$\therefore \lim\limits_{n\to\infty}a_n\ \exists$

型三 中值定理求极限

1. $\lim\limits_{n\to\infty} n^2[\ln(1+\frac{1}{n})-\ln(1+\frac{1}{n+1})]$

解：令 $f(x)=\ln(1+x)$. $f'(x)=\frac{1}{1+x}$

$[\cdots]=f(\frac{1}{n})-f(\frac{1}{n+1})=f'(\xi)(\frac{1}{n}-\frac{1}{n+1})=\frac{1}{n(n+1)}\cdot\frac{1}{1+\xi}\ (\frac{1}{n+1}<\xi<\frac{1}{n})$

原 $=\lim\limits_{n\to\infty}\frac{n}{n+1}\cdot\frac{1}{1+\xi}=1$

2. $f'(0)=0$, $f''(0)=\pi$. $f(x)$ = 阶可导.

求 $\lim\limits_{x\to0}\frac{f(x)-f[\ln(1+x)]}{x^3}$

解：$f(x)-f[\ln(1+x)]=f'(\xi)[x-\ln(x+1)]\quad(\ln(1+x)<\xi<x)$

原 $=\lim\limits_{x\to0}\frac{x-\ln(1+x)}{x^2}\cdot\frac{f'(\xi)}{x}$

$=\frac{1}{2}\lim\limits_{x\to0}\frac{f'(\xi)-f'(0)}{\xi-0}\times\frac{\xi}{x}$

$=\frac{\pi}{2}\lim\limits_{x\to0}\frac{\xi}{x}$

$\because \ln(1+x)<\xi<x$

$\therefore$ ① $x<0$ $\quad\frac{\ln(1+x)}{x}>\frac{\xi}{x}>1 \Rightarrow \lim\limits_{x\to0^-}\frac{\xi}{x}=1$

② $x>0$ $\quad\frac{\ln(1+x)}{x}<\frac{\xi}{x}<1 \Rightarrow \lim\limits_{x\to0^+}\frac{\xi}{x}=1$

$\therefore \lim\limits_{x\to0}\frac{\xi}{x}=1$ $\quad\therefore$ 原式 $=\frac{\pi}{2}$

型四 不定型极限

P13 例1. 解：

(4) 原 $=\lim\limits_{x\to0}\frac{\arctan x-x}{x^3}+\lim\limits_{x\to0}\frac{x-\arcsin x}{x^3}$

$=-\frac{1}{3}-\lim\limits_{x\to0}\frac{(1-x^2)^{-\frac{1}{2}}-1}{3x^2}$

$=-\frac{1}{3}-\lim\limits_{x\to0}\frac{-\frac{1}{2}(-x^2)}{3x^2}=-\frac{1}{3}-\frac{1}{6}=-\frac{1}{2}$

(5) 解：原 $=\lim\limits_{x\to0}e^{\sin^2x}\times\frac{e^{x^2-\sin^2x}-1}{x^4}$

$=\lim\limits_{x\to0}\frac{x+\sin x}{x}\times\frac{x-\sin x}{x^3}$

$=2\times\lim\limits_{x\to0}\frac{1-\cos x}{3x^2}=\frac{1}{3}$

$(=\lim\limits_{x\to0}\frac{2x-\sin2x}{4x^3}=2\lim\limits_{x\to0}\frac{2x-\sin2x}{(2x)^3}=2\lim\limits_{t\to0}\frac{t-\sin t}{t^3})$

2. $\lim\limits_{x\to0}\frac{\sin6x+xf(x)}{x^3}=0$, $\lim\limits_{x\to0}\frac{6+f(x)}{x^2}$?

解：$0=\lim\limits_{x\to0}\frac{\sin6x-6x}{x^3}+\lim\limits_{x\to0}\frac{6+f(x)}{x^2}$

$\lim\limits_{x\to0}\frac{\sin6x-6x}{x^3}=6^3\cdot\lim\limits_{t\to0}\frac{\sin t-t}{t^3}=6^3\lim\limits_{t\to0}\frac{\cos t-1}{3t^2}=-36$

$\therefore\lim\limits_{x\to0}\frac{6+f(x)}{x^2}=36$

3. $\lim\limits_{x\to0}\dfrac{1-\cos x\cdot\cos2x\cdot\cos3x\cdots\cos nx}{x^2}$

$=\lim\limits_{x\to0}\dfrac{1-\cos x}{x^2}+\lim\limits_{x\to0}\dfrac{\cos x-\cos x\cos2x}{x^2}+\cdots+\lim\limits_{x\to0}\dfrac{\cos x\cdots\cos(n-1)x-\cos x\cdots\cos nx}{x^2}$

$=\dfrac12+\lim\limits_{x\to0}\cos x\cdot\dfrac{1-\cos2x}{x^2}+\cdots+\lim\limits_{x\to0}\cos x\cdots\cos(n-1)x\,\dfrac{1-\cos nx}{x^2}$

$=\dfrac12+\lim\limits_{x\to0}\dfrac{1-\cos2x}{x^2}+\cdots+\lim\limits_{x\to0}\dfrac{1-\cos nx}{x^2}$

$=\dfrac12+\dfrac42+\dfrac92+\cdots+\dfrac{n^2}{2}$

4. $\lim\limits_{x\to\infty}\left[\dfrac{x^2+3x+1}{x+1}e^{\frac1x}-x\right]$ $(\infty-\infty)$

$=\lim\limits_{x\to\infty}\left[\dfrac{x^2+3x+1}{x+1}\cdot(e^{\frac1x}-1)+\dfrac{x^2+3x+1}{x+1}-x\right]$

$=\lim\limits_{x\to\infty}\dfrac{x^2+3x+1}{x+1}(e^{\frac1x}-1)+\lim\limits_{x\to\infty}\dfrac{2x+1}{x+1}$

$=\lim\limits_{x\to\infty}\dfrac{x^2+3x+1}{x^2+x}+2=3$

5. $\lim\limits_{x\to+\infty}\dfrac{(1+\frac1x)^{x^2}}{e^x}$ $\left(\dfrac{\infty}{\infty}\right)$

$=\lim\limits_{x\to+\infty}e^{x^2\ln(1+\frac1x)-x}$

$=e^{\lim\limits_{x\to+\infty}\frac{\ln(1+\frac1x)-x}{\frac{1}{x^2}}}=e^{\lim\limits_{t\to0}\frac{\ln(1+t)-t}{t^2}}=e^{-\frac12}$

说明: $\lim\square^{\square}=a^{\lim\square}$ 都有极限才可用
$\square\to a$

$\lim\limits_{x\to+\infty}\left[\dfrac{(1+\frac1x)^x}{e}\right]^x=1$ X

$\lim\square^{\square}$

6. $\lim\limits_{x\to1}\dfrac{(x-1)\ln x}{x-1-\ln x}=\lim\limits_{x\to1}\dfrac{(x-1)\ln[1+(x-1)]}{(x-1)-\ln[1+(x-1)]}\xlongequal{x-1=t}\lim\limits_{t\to0}\dfrac{t\ln(1+t)}{t-\ln(1+t)}$

$=\lim\limits_{t\to0}\dfrac{t^2}{x-\ln(1+t)}=2$

7. $\lim\limits_{x\to-\infty}(\sqrt{4x^2-8x+5}+2x)$

$=\lim\limits_{x\to-\infty}\dfrac{-8x+5}{\sqrt{4x^2-8x+5}-2x}=2$

8 P86. 例6.

(5) 原式 $=-\lim\limits_{x\to0^+}x\ln x$

$=-\lim\limits_{x\to0^+}\ln x^x$

$=0$

$\ln(1-x)\sim-x$

$\lim\limits_{x\to0^+}x^x=1$

(6) 原式 $= e^{\lim_{x\to0^+}\tan x\cdot\ln x^{-\frac{1}{2}}}$ ∞^0

$= e^{-\frac{1}{2}\lim_{x\to0^+}x\ln x} = e^{-\frac{1}{2}\lim_{x\to0^+}\ln x^x} = 1$

P23 题型五 变积分限函数极限

例1. 解: $\lim_{x\to0}\frac{x-\tan x}{x^3} = -\frac{1}{3}\lim_{x\to0}\frac{\tan^2x}{x^2} = -\frac{1}{3}$

$\Rightarrow x-\tan x \sim -\frac{1}{3}x^3$

$\sqrt{x+1}-1 = (1+x)^{\frac{1}{2}}-1 \sim \frac{1}{2}x$

$\therefore$ 原式 $= -6\lim_{x\to0}\frac{\int_0^x e^t\cos t\,dt - x - \frac{x^2}{2}}{x^4}$

$= -\frac{3}{2}\lim_{x\to0}\frac{e^x\cos x-1-x}{x^3}$

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+o(x^3)$

$\cos x = 1-\frac{x^2}{2!}+o(x^3)$

$e^x\cos x = 1+x+(-\frac{1}{2}+\frac{1}{6})x^3+o(x^3)$

$e^x\cos x-1-x \sim -\frac{1}{3}x^3$

$\therefore$ 原式 $= -\frac{3}{2}\times(-\frac{1}{3}) = \frac{1}{2}$

例2. 解: $\int_1^{\frac{1}{x}} f(xt)\,dt = \frac{1}{x}\int_1^{\frac{1}{x}} f(xt)\,d(xt) = \frac{1}{x}\int_x^1 f(t)\,dt$

$= -\frac{1}{x}\int_1^x f(t)\,dt$

原 $= \lim_{x\to1}\frac{-\frac{1}{x}\int_1^x f(t)\,dt}{\cancel{(x+1)}(x-1)(x^2+x+1)}$

$= -\frac{1}{3}\lim_{x\to1}\frac{\int_1^x f(t)\,dt}{x-1}$

$= -\frac{1}{3}\lim_{x\to1}f(x) = -\frac{1}{3}$

例3. 解

$\int_0^x tf(x^2-t^2)\,dt = -\frac{1}{2}\int_0^x f(x^2-t^2)\,d(x^2-t^2) = -\frac{1}{2}\int_{x^2}^0 f(u)\,du$

$= \frac{1}{2}\int_0^{x^2} f(u)\,du$

$\int_0^x f(x-t)\,dt = -\int_0^x f(x-t)\,d(x-t) = -\int_x^0 f(u)\,du = \int_0^x f(u)\,du$

原 $= \lim_{x\to0}\frac{xf(x^2)}{2x\int_0^x f(u)du + x^2f(x)}$

$= \lim_{x\to0}\frac{f(x^2)}{2\int_0^x f(u)du + xf(x)}$

$\cancel{= \lim_{x\to0}\frac{f(x^2)-f(0)}{x^2}}$

$= \lim_{x\to0}\frac{\frac{f(x^2)-f(0)}{x^2}}{\frac{2\int_0^x f(u)du}{x^2}+\frac{f(x)-f(0)}{x}}$

$\lim_{x\to0}\frac{f(x^2)-f(0)}{x^2} = f'(0) = 2$

$\lim_{x\to0}\frac{\int_0^x f(u)du}{x^2} = \lim_{x\to0}\frac{f(x)-f(0)}{2x} = 1$ $\therefore$ 原 $= \frac{2}{2\times1+2} = \frac{1}{2}$

例4. 解: $\int_0^x f(x-t)dt = -\int_0^x f(x-t)d(x-t) = \int_0^x f(u)du$

原 $=\lim\limits_{x\to0}\left(\cos x+\int_0^x f(u)du\right)^{\frac{1}{x^2}}$

$=\lim\limits_{x\to0}\left\{\left[1+\underbrace{\left(\cos x-1+\int_0^x f(u)du\right)}_{\Delta}\right]^{\frac{1}{\Delta}}\right\}^{\frac{\Delta}{x^2}}$

$=e^{\lim\limits_{x\to0}\frac{\cos x-1+\int_0^x f(u)du}{x^2}} = e^{-\frac{1}{2}+\frac{1}{2}\lim\limits_{x\to0}\frac{f(x)-f(0)}{x}}$

例5. $f(x)$连续 且 ① $f(0)\neq0$；② $f(0)=0$, $f'(0)\neq0$

求 $\lim\limits_{x\to0}\frac{x\int_0^x f(x-t)dt}{\int_0^x t f(x-t)dt}$

解: $\int_0^x f(x-t)dt = \int_0^x f(u)du$

$\int_0^x t f(x-t)dt \xlongequal{令x-t=u} \int_x^0 (x-u)f(u)(-du) = \int_0^x (x-u)f(u)du$

$= x\int_0^x f(u)du - \int_0^x u f(u)du$

原式 $=\lim\limits_{x\to0}\frac{\int_0^x f(u)du + xf(x)}{\int_0^x f(u)du}$

① $f(0)\neq0$

原 $=\lim\limits_{x\to0}\frac{\frac{\int_0^x f(u)du}{x}+f(x)}{\frac{\int_0^x f(u)du}{x}} = \frac{f(0)+f(0)}{f(0)} = 2$

② $f(0)=0$ $f'(0)\neq0$

原 $=\lim\limits_{x\to0}\frac{\frac{\int_0^x f(u)du}{x^2}+\frac{f(x)-f(0)}{x}}{\frac{\int_0^x f(u)du}{x^2}}$

$\because \lim\limits_{x\to0}\frac{\int_0^x f(u)du}{x^2} = \frac{1}{2}\lim\limits_{x\to0}\frac{f(x)-f(0)}{x} = \frac{1}{2}f'(0)$

$\therefore$ 原式 $=\frac{\frac{1}{2}f'(0)+f'(0)}{\frac{1}{2}f'(0)} = 3.$

型六 间断点

例1. $f(x)=\frac{\ln|x|}{x^2-1}e^{\frac{1}{x-2}}$

解: $x=-1,\ 0,\ 1,\ 2$ 为间断点

$\lim\limits_{x\to-1}f(x)=\lim\limits_{x\to-1}\frac{e^{\frac{1}{x-2}}}{x-1}\cdot\frac{\ln(-x)}{x+1} = -\frac{1}{2}e^{-\frac{1}{3}}\lim\limits_{x\to-1}\frac{\ln[1-(x+1)]}{x+1} = \frac{1}{2}e^{-\frac{1}{3}}$

$\therefore x=-1$ 为可去间断点.

$\lim\limits_{x\to0}f(x)=+\infty \Rightarrow x=0$ 为第二类间断点

$\lim\limits_{x\to1}f(x)=\lim\limits_{x\to1}\frac{e^{\frac{1}{x-2}}}{x+1}\cdot\frac{\ln x}{x-1} = \frac{1}{2}e^{-1}\lim\limits_{x\to1}\frac{\ln[1+(x-1)]}{x-1} = \frac{1}{2}e^{-1}$

$\Rightarrow x=1$ 为可去间断点

$f(2-0)=0$, $f(2+0)=+\infty$ $\Rightarrow$ $x=2$为第二类间断点

例2. $f(x)=\dfrac{2-2^{\frac{1}{x-1}}}{1+2^{\frac{1}{x-1}}}$

解: $f(1-0)=2$ 与 $f\overset{1+0}{(+\infty)}=0$

例3. P25. 例5.

(1) $f(x)=\lim\limits_{t\to x}\left(\frac{\sin x}{\sin t}\right)^{\frac{t}{\sin x-\sin t}}=\lim\limits_{t\to x}\left[\left(1+\frac{\sin x-\sin t}{\sin t}\right)^{\frac{\sin t}{\sin x-\sin t}}\right]^{\frac{t}{\sin t}}=e^{\frac{x}{\sin x}}$

(2) $x=k\pi$ $(k\in Z)$ 为间断点

$\lim\limits_{x\to 0}f(x)=e$ $\Rightarrow$ $x=0$为可去间断点.

$f(\pi-0)=+\infty$, $f(\pi+0)=0$ $\Rightarrow$ $x=\pi$为第二类间断点

同理 $x=k\pi$ ($k\in Z$, 且 $k\neq 0$)为第二类间断点.

第二模块 微分学

Part I 一元微分学

一 defs.

1. 导数 — $f'(a) \triangleq \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

Notes:

① $f'(a) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$

② $f'_-(a) = \lim\limits_{\Delta x \to 0^-} \frac{\Delta y}{\Delta x} \left(= \lim\limits_{x \to a^-} \frac{f(x)-f(a)}{x-a}\right)$

$f'_+(a) = \lim\limits_{\Delta x \to 0^+} \frac{\Delta y}{\Delta x} = \left(\lim\limits_{x \to a^+} \frac{f(x)-f(a)}{x-a}\right)$

$f'(a)\exists \Leftrightarrow f'_-(a), f'_+(a) \exists$ 且相等.

例1. $f(x) = \begin{cases} \frac{x\, 2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $f'(0)$?

解: $\lim\limits_{x \to 0} \frac{f(x)-f(0)}{x} = \lim\limits_{x \to 0} \frac{2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}$

$f'_-(0) = 0 \neq f'_+(0) = 1$ ∴ $f'(0)$ 不存在.

例2. $\forall x,\ f(x+1) = k f(x)$

$x \in [0,1]$ 时, $f(x) = x(1-x^2)$. 已知 $f'(0)\exists$, 求 k

解: 1° $x \in [-1, 0]$ 时, $x+1 \in [0,1]$

$f(x) = \frac{1}{k} f(x+1) = \frac{1}{k}(x+1)(-x^2-2x) = -\frac{1}{k}(x+1)\cdot x \cdot (x+2)$

2° $f'_-(0) = \lim\limits_{x \to 0^-} \frac{f(x)}{x} = -\frac{2}{k}$

$f'_+(0) = \lim\limits_{x \to 0^+} \frac{f(x)}{x} = 1$

∵ $f'(0)\exists$, ∴ $k = -2$

③. $f(x)$连续 $\lim\limits_{x \to a} \frac{f(x)-b}{x-a} = A \Rightarrow f(a) = b,\ f'(a) = A$

④ 求导改变奇偶性

如: $f(x)$ 可导, 奇函数, $f'(1) = 2 \Rightarrow \begin{cases} f'(0) = 0 \\ f'(-1) = 2 \end{cases}$

⑤ 定义判断可导性
- 保双侧
- 不可拆
- 阶相同

保双侧 $\Delta x\to 0\begin{cases}\to 0^- \\ \to 0^+\end{cases}$ $\left(x\to a\begin{cases}\to a^- \\ \to a^+\end{cases}\right)$.

如：$\lim\limits_{h\to 0}\frac{f(1-\cos h)-f(0)}{h^2}$ ∃ $\overset{?}{\Rightarrow}$ $f'(0)$ ∃ X

$1-\cos h\to 0^+$

||

$\lim\limits_{h\to 0}\frac{f[0+(1-\cos h)]-f(0)}{1-\cos h}\cdot\frac{1-\cos h}{h^2}=\frac{1}{2}f'_+(0)$

不可跨：$\Delta y=f(a+\square)-f(a)$

必须 a

例3. ① 设 $f'(a)$∃.

$\lim\limits_{h\to 0}\frac{f(a+2h)-f(a-h)}{h}$?

解：原 $=\lim\limits_{h\to 0}\left[2\frac{f(a+2h)-f(a)}{2h}+\frac{f(a-h)-f(a)}{-h}\right]=3f'(a)$

② If $\lim\limits_{h\to 0}\frac{f(a+h)-f(a-h)}{h}$ ∃ $\overset{?}{\Rightarrow}$ $f'(a)$∃ X

反例：$f(x)=\begin{cases}3(x-1), & x\neq 1\\ 2, & x=1\end{cases}$ 取 $a=1$.

$\lim\limits_{h\to 0}\frac{f(1+h)-f(1-h)}{h}=\lim\limits_{h\to 0}\frac{3h-(-3h)}{h}=6$ ∃

而 $\lim\limits_{x\to 1}f(x)=0\neq f(1)=2$ ⇒ $f(x)$在$x=1$不连续 ⇒ $f(x)$在$x=1$不可导

阶相同：$\frac{f(a+\Delta_1)-f(a)}{\Delta_2}$, $\Delta_2=O(\Delta_1)$

2. 可微 — $\Delta y=f(a+\Delta x)-f(a)$ $(=f(x)-f(a))$

$\overset{\text{if}}{=}A\Delta x+o(\Delta x)$ 称$f(x)$在$x=a$可微.

$A\,dx=A\Delta x\triangleq dy|_{x=a}$

Notes:

① 可导 ⇔ 可微.

"⇒" $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(a)$ ⇒ $\frac{\Delta y}{\Delta x}=f'(a)+\alpha$, $\alpha\to 0(\Delta x\to 0)$.

⇒ $\Delta y=\underline{f'(a)\Delta x}+\alpha\Delta x$

∵ $\lim\limits_{\Delta x\to 0}\frac{\alpha\Delta x}{\Delta x}=0$ ∴ $\alpha\Delta x=o(\Delta x)$

$\Delta y=f'(a)\Delta x+o(\Delta x)$

"⇐" $\Delta y=A\Delta x+o(\Delta x)$

$\frac{\Delta y}{\Delta x}=A+\frac{o(\Delta x)}{\Delta x}$

$\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=A$ ⇒ $f'(a)=A$

② $A=f'(0)$

例1. $y=y(x)$: $\Delta y=\frac{xy}{1+x^2}\Delta x+o(\Delta x)$ $y(0)=1$, $y(x)$?

解: $\Delta y=\frac{xy}{1+x^2}\Delta x+o(\Delta x)$

$\Downarrow$

$\frac{dy}{dx}=\frac{xy}{1+x^2}\Rightarrow\frac{dy}{dx}-\frac{x}{1+x^2}y=0$

$y=Ce^{-\int-\frac{x}{1+x^2}dx}=C\sqrt{1+x^2}$

$\because y(0)=1$ $\therefore C=1$ $y(x)=\sqrt{1+x^2}$

例2. $f(u)$ 可导. $y=f(x^2)$. $x_0=-1$.

当 $\Delta x=0.001$ 时 Δy 的线性部分为 0.05, 求 $f'(1)$

解: $dy|_{x=-1}=y'(-1)\Delta x=0.05$

$y'(x)=2xf'(x^2)$, $y'(-1)=-2f'(1)$

$-2f'(1)\times0.001=0.05$

不可导:

如: $f(x)=|x|$. $x=0$ 处连续但不可导

$a=0$. $\lim\limits_{h\to0}\frac{f(0+h)-f(0-h)}{h}=\lim\limits_{h\to0}\frac{|h|-|-h|}{h}=0$

见 $f(x+y)$ —— 导数定义

例3. $\forall x,y$ $f(x+y)=f(x)\cdot f(y)$ $f'(0)=1$. 求 $f(x)$.

解: $f'(x)=\lim\limits_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim\limits_{h\to0}\frac{f(x)f(h)-f(x)}{h}=f(x)\lim\limits_{h\to0}\frac{f(h)-1}{h}$

$\because f(0)=f^2(0)$ $\therefore f(0)=0$ 或 $f(0)=1$

If $f(0)=0$. $\forall x$, $f(x)=f(x+0)=f(x)f(0)=0$

即 $f(x)\equiv0$ 矛盾 $\therefore f(0)=1$

$\therefore f'(x)=f(x)\lim\limits_{h\to0}\frac{f(h)-f(0)}{h}=f(x)$ 即 $\frac{dy}{dx}-y=0$

$y=Ce^{-\int-dx}=Ce^x$

$\because f(0)=1$ $\therefore C=1$ $\therefore f(x)=e^x$

二、工具.

(一) 公式

① $(\sin x)^{(n)}=\sin(x+\frac{n\pi}{2})$

② $(\cos x)^{(n)}=\cos(x+\frac{n\pi}{2})$

③ $(\frac{1}{ax+b})^{(n)}=\frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$

④ $(uv)^{(n)}=C_n^0u^{(n)}v+C_n^1u^{(n-1)}v'+\dots+C_n^nuv^{(n)}$

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(二) 四则

例4 $f(x)=x(x-1)(x+2)\cdots(x-99)(x+100)$ $f'(0)$?

解: 法一: $f'(x)=(x-1)(x+2)\cdots(x+100)+x(x+2)\cdots(x+100)$
$+\cdots+x\cdots(x-99)$

$$f'(0)=(-1)\cdot 2\cdot(-3)\cdots(-99)\cdot 100=100!$$

法二: $f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}(x-1)(x+2)\cdots(x+100)=100!$

(三) 复合

(四) 反函数

$y=f(x)\Rightarrow x=\varphi(y)$

$y=\ln(x+\sqrt{x^2+1})$ 求反函数.

$$\begin{cases}x+\sqrt{x^2+1}=e^{y}\\-x+\sqrt{x^2+1}=e^{-y}\end{cases}\Rightarrow x=\frac{e^{y}-e^{-y}}{2}$$

Th1. $y=f(x)$: $f'(x)\neq 0$ $x=\varphi(y)$

$$\varphi'(y)=\frac{1}{f'(x)}$$

Th2. $\varphi''(y)=\dfrac{d\varphi'(y)}{dy}=\dfrac{d\left[\frac{1}{f'(x)}\right]/dx}{dy/dx}=\dfrac{1}{f'(x)}\cdot\left[-\dfrac{1}{f'^2(x)}\right]\cdot f''(x)=-\dfrac{f''(x)}{f'^3(x)}$

题型一. 概念

1. P38 例4. 解.

$\lim\limits_{h\to 0}\dfrac{f[0+(1-\cos h)]-f(0)}{h^2}$ × $\quad 1-\cos h\to 0^+$

$\lim\limits_{h\to 0}\dfrac{f[0+(1-e^h)]-f(0)}{h}$ ✓ $\quad 1-e^h\begin{cases}\to 0^-, & h\to 0^+\\ \to 0^+, & h\to 0^-\end{cases}$

$\lim\limits_{h\to 0}\dfrac{f[0+(h-\sin h)]-f(0)}{h^2}$

$\lim\limits_{h\to 0}\dfrac{f(0+2h)-f(0+h)}{h}$

2. P4. 例5. 解:

(A) $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=A\Rightarrow f(0)=0,\ f'(0)=A$

(B) $2f(0)=0$

(D) $\lim\limits_{x\to 0}\dfrac{f(0+x)-f(0-x)}{x}$

3. $f(x)=x\sqrt{|x|}\cdot|x^2-1|$. 讨论不可导点

解.（$f'(0)=0$）

$\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\sqrt{|x|}\cdot|x^2-1|=0 \Rightarrow f'(0)=0.$

$\lim\limits_{x\to -1}\frac{f(x)-f(-1)}{x+1}=\lim\limits_{x\to -1}x\sqrt{|x|}\cdot|x-1|\frac{|x+1|}{x+1}=-2\lim\limits_{x\to -1}\frac{|x+1|}{x+1}$

$f'_-(-1)=2 \neq f'_+(-1)=-2$

$\lim\limits_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1}x\sqrt{|x|}\,|x+1|\frac{|x-1|}{x-1}=2\lim\limits_{x\to 1}\frac{|x-1|}{x-1}$

$f'_-(1)=-2 \neq f'_+(1)=2.$

Note：$f(x)-f(a)$ 表达式中含 $|x-a|^{\alpha}$

$$\begin{cases}\alpha>1, & f'(a)=0\\ \alpha\leq 1, & f'(a)\text{ 不存在.}\end{cases}$$

如：$f(x)=\underbrace{|x^3-x|}_{\alpha\leq 1}\cdot\sqrt[3]{x+1}=|x|\cdot|x-1|\cdot|x+1|\cdot(x+1)^{\frac{1}{3}}$

$x=-1$，$f'(-1)=0$

$x=0$，$f'(0)$ 无

$x=1$，$f'(1)$ 无

题二 隐函数求导

1. $2\ln(y-x)+2xy=e^{y-1}-1$，求 $dy|_{x=0}$.

解. 1° $x=0$ 时，$\ln y=e^{y-1} \Rightarrow y=1$

2° $2\frac{y'-1}{y-x}+2y+2xy'=e^{y-1}y'$

把 $x=0$，$y=1$ 代入 $2y'(0)-2+2+0=y'(0)$，$y'(0)=0$

3° $dy|_{x=0}=0dx$

2. P37 例4 解：

$\int_x^{x+x^2+y}e^{-(t-x)^2}dt=xy$

1° 左 $=\int_x^{x+x^2+y}e^{-(t-x)^2}d(t-x)=\int_0^{x^2+y}e^{-u^2}du$

2° $\int_0^{x^2+y}e^{-u^2}du=xy \Rightarrow e^{-(x^2+y)^2}(2x+y')=y+xy'$

P37 例5 解：

1° $x=0 \Rightarrow y=1$

2° $e^y+xe^yy'+y'e^x+ye^x=y+xy'$　　$y'(0)=-e$

3° $2e^yy'+xy''e^y+xy'^2e^y+y''e^x+2y'e^x+ye^x=2y'+xy''$

$-2e^2+y''(0)-2e+1=-2e$　　$y''(0)=2e^2-1$

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题型三、分段函数求导

1. $f(x)=\int_0^1 |x^2-t^2|dt \quad (x>0)$，求 $f'(x)$，求 m

解：1° $0<x<1$ 时

$f(x)=\int_0^x (x^2-t^2)dt+\int_x^1 (t^2-x^2)dt$

$=\frac{2}{3}x^3+\frac{1-x^3}{3}-x^2(1-x)=\frac{4}{3}x^3-x^2+\frac{1}{3}$；

$x\geq 1$ 时

$f(x)=\int_0^1 (x^2-t^2)dt=x^2-\frac{1}{3}$

$$f(x)=\begin{cases}\frac{4}{3}x^3-x^2+\frac{1}{3}, & 0<x<1\\ x^2-\frac{1}{3}, & x\geq 1\end{cases}$$

2° $0<x<1$ 时 $f'(x)=4x^2-2x$；

$x>1$ 时 $f'(x)=2x$；

$f'_-(1)=\lim\limits_{x\to 1^-}\frac{\frac{4}{3}x^3-x^2-\frac{2}{3}}{x-1}=\lim\limits_{x\to 1^-}(4x^2-2x)=2$；

$f'_+(1)=\lim\limits_{x\to 1^+}\frac{x^2-1}{x-1}=2$.

$\Rightarrow f'(1)=2$

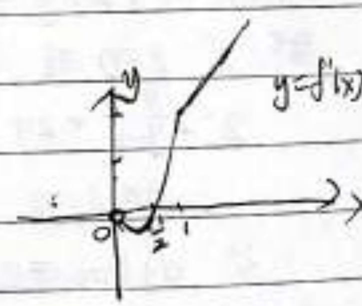
$$f'(x)=\begin{cases}4x^2-2x, & 0<x<1\\ 2x, & x\geq 1\end{cases}$$

3° ∵ $0<x<\frac{1}{2}$ 时，$f'(x)<0$，

$x>\frac{1}{2}$ 时，$f'(x)>0$，

∴ $x=\frac{1}{2}$ 为最小点

$m=f(\frac{1}{2})$



2. $f(x)=\begin{cases}\ln(1+2x), & x>0\\ ax^2+bx+c, & x\leq 0\end{cases}$ $f''(0)$ ∃，求 a, b, c

解：$f(0+0)=0=f(0)=f(0-0)=c \Rightarrow c=0$

$$f'(x)=\begin{cases}\frac{2}{1+2x}, & x>0\\ 2ax+b, & x\leq 0\end{cases}$$

∵ $f'(x)$ 在 $x=0$ 连续　∴ $b=2$

$$\therefore f'(x)=\begin{cases}\frac{2}{1+2x}, & x>0\\ 2ax+2, & x\leq 0\end{cases}$$

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$$f_+''(0) = \lim_{x\to0^+} \frac{\frac{2}{1+2x}-2}{x} = \lim_{x\to0^+}\frac{\frac{-4x}{1+2x}}{x} = -4$$

$$f_-''(0) = \lim_{x\to0^-}\frac{2ax+2-2}{x} = 2a.$$

$$\therefore\ a=-2$$

型四 高阶导数

✓ ① 归纳

② 公式 $(uv)^{(n)}$

③ 泰氏

例1 $f(x)=\frac{5x-1}{x^2-x-2}$ 　$f^{(n)}(x)$

解：$f(x)=\frac{5x-1}{(x+1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-2}$

$$A(x-2)+B(x+1)=5x-1 \Rightarrow \begin{cases}A+B=5\\ -2A+B=-1\end{cases} \Rightarrow \begin{cases}A=2\\ B=3\end{cases}$$

$$f(x)=2\cdot\frac{1}{x+1}+3\cdot\frac{1}{x-2}$$

$$f^{(n)}(x)=2\cdot\frac{(-1)^n n!}{(x+1)^{n+1}}+3\cdot\frac{(-1)^n n!}{(x-2)^{n+1}}$$

2. $f(x)=\ln(1-x-2x^2)$ 求 $f^{(n)}(x)$

解：$f(x)=\ln(1+x)(1-2x) = \ln(1+x)+\ln(1-2x)$

$$f'(x)=\frac{1}{1+x}+\frac{2}{2x-1}$$

$$f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{(x+1)^n}+\frac{(-1)^{n-1}(n-1)!\cdot 2^n}{(2x-1)^n}$$

---

3. $f(x)=(1+2x^2)\ln(1+x^2)$ 　$f^{(20)}(0)$

解．$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^9}{9}-\frac{x^{10}}{10}+o(x^{10})$

$$\ln(1+x^2)=x^2-\cdots+\frac{x^{18}}{9}-\frac{x^{20}}{10}+o(x^{20})$$

$$f(x)=\cdots+\left(-\frac{1}{10}+\frac{2}{9}\right)x^{20}+o(x^{20})$$

$$=f(0)+\cdots+\frac{f^{(20)}(0)}{20!}x^{20}+o(x^{20})$$

$$\frac{f^{(20)}(0)}{20!}=-\frac{1}{10}+\frac{2}{9}$$

Part Ⅱ 一 一元微分学定理

一、中值定理

Th1 (Rolle)

证：$f(x) \in C[a,b] \Rightarrow \exists\, m, M$

Case1. $m=M$. $f(x)\equiv C_0$, $\forall \xi\in(a,b)$, $f'(\xi)=0$;

Case2. $m<M$

$\because f(a)=f(b)$

$\therefore m, M$ 至少一个在(a,b)内.

设 $\exists\, \xi\in(a,b)$, $f(\xi)=M \Rightarrow f'(\xi)=0$ 或不存在

$\because f(x)$在(a,b)内可导 $f'(\xi)=0$

Th2 (Lagrange)

$$f'(\xi)=\frac{f(b)-f(a)}{b-a} \Longleftrightarrow f(b)-f(a)=f'(\xi)(b-a)$$

$$\Longleftrightarrow f(b)-f(a)=f'[a+\theta(b-a)](b-a) \quad (0<\theta<1)$$

例1. 设 $f(x)\in C[a,b]$, (a,b)内 $f'(x)\equiv 0$.

证明 $f(x)\equiv C_0$.

证：$\forall\, x_1, x_2\in[a,b]$ 且 $x_1\neq x_2$.

$f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)$ （ξ在x_1与x_2内）

$\because f'(x)\equiv 0 \quad (a<x<b)$

$\therefore f(x_1)=f(x_2)$

$\therefore f(x)\equiv C_0 \quad (a\le x\le b)$

Th3. 柯西

Th4. (Taylor).

背景：$\lim\limits_{x\to 0}\frac{x-\sin x}{x^3} \neq \lim\limits_{x\to 0}\frac{x-x}{x^3}=0$

条件：$f(x)$在 $x=x_0$ 邻域内 $n+1$ 阶可导. 则

$$f(x)=P_n(x)+R_n(x)$$

其中 $P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

推广：条件：$f(x)$ 在 $x=x_0$ 邻域内 n阶连续可导

结论：$f(x)=P_n(x)+o((x-x_0)^n)$

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$$R_n(x)\begin{cases}\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} & \text{— L}\\ o((x-x_0)^n) & \text{— P 皮亚诺}\end{cases}$$

If $x_0=0$

$$f(x)=f(0)+f'(0)x+\cdots+\frac{f^{(n)}(0)}{n!}x^n+R_n(x)$$

型一 关于 θ

① $f(b)-f(a)=f'[a+\theta(b-a)](b-a)\quad(0<\theta<1)$

② $f(x)=P_n(x)+\dfrac{f^{(n+1)}[x_0+\theta(x-x_0)]}{(n+1)!}(x-x_0)^{n+1}\quad(0<\theta<1)$ ——→ 我认为X抽象时使用

③ $\int_a^b f(x)dx=f(\xi)(b-a)=f[a+\theta(b-a)](b-a)\quad(0\le\theta\le1)$

Notes:

① If $f(x)$ 表达式具体，求出 θ.

② If $f(x)$ 抽象，不求 θ

例1. $f(x)=\arctan x$，$a\neq0$，$f(a)=f'(\theta a)\cdot a$ 求 $\lim\limits_{a\to0}\theta$.

解：$f'(x)=\dfrac{1}{1+x^2}$

$$f(a)=f(a)-f(0)=f'[0+\theta(a-0)]a\quad(0<\theta<1)$$

$$\frac{\arctan a}{a}=\frac{1}{1+\theta^2a^2}\Rightarrow\theta^2=\frac{a-\arctan a}{a^2\arctan a}$$

$$\lim_{a\to0}\theta^2=\lim_{a\to0}\frac{1-\frac{1}{1+a^2}}{3a^2}=\frac13\lim_{a\to0}\frac{\frac{a^2}{1+a^2}}{a^2}=\frac13$$

$$\lim_{a\to0}\theta=\frac{1}{\sqrt3}$$

2. $\int_0^x e^t dt=xe^{\theta x}$，求 $\lim\limits_{x\to0}\theta$

解：$xe^{\theta x}=e^x-1\Rightarrow e^{\theta x}=\dfrac{e^x-1}{x}$

$$\theta=\frac1x\ln\frac{e^x-1}{x}$$

$$\lim_{x\to0}\theta=\lim_{x\to0}\frac{\ln\left(1+\frac{e^x-1-x}{x}\right)}{x}=\lim_{x\to0}\frac{e^x-1-x}{x^2}=\lim_{x\to0}\frac{e^x-1}{2x}=\frac12$$

3. $f(x)$ 二阶连续可导，$f''(a)\neq0$. $f(a+h)=f(a)+f'(a+\theta h)h\quad(0<\theta<1)$

求 $\lim\limits_{h\to0}\theta$?

解：$f(a+h)=f(a)+f'(a)h+\dfrac{f''(a)}{2!}h^2+o(h^2)$

$$\Rightarrow f'(a+\theta h)=f'(a)+\frac12f''(a)h+o(h)$$

$$\Rightarrow\theta\,\frac{f'(a+\theta h)-f'(a)}{\theta h}=\frac12f''(a)+\frac{o(h)}{h}$$

$$\Rightarrow\lim_{h\to0}\theta\times f''(a)=\frac12f''(a)$$

$\because f''(a)\neq0$，$\therefore\lim\limits_{h\to0}\theta=\dfrac12$

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型二 $f^{(n)}(\xi)=0$ (Rolle)

Q. $f''(\xi)=0$ $\begin{cases} f(a)=f(c)=f(b) \\ f'(\xi_1)=f'(\xi_2) \end{cases}$

例1. $f(x)$ 二阶可导，$\lim\limits_{x\to 0}\frac{f(x)}{x}=1$，$f(1)=1$

证 $\exists\xi\in(0,1)$，$f''(\xi)=0$

证：$\lim\limits_{x\to 0}\frac{f(x)}{x}=1 \Rightarrow f(0)=0 \quad f'(0)=1$

$\exists c\in(0,1)$，使 $f'(c)=\frac{f(1)-f(0)}{1-0}=1$

$\because f'(0)=f'(c)=1$

$\therefore \exists\xi\in(0,c)\subset(0,1)$，使 $f''(\xi)=0$.

例2. $f(x)\in C[0,4]$，在 $(0,4)$ 内二阶可导

$2f(0)=f(1)+f(2)=\int_2^4 f(x)\,dx$

证：$\exists\xi\in(0,1)$，$f''(\xi)=0$

证：1° $f(x)\in C[1,2] \Rightarrow \exists\, m, M$

$$m\le\frac{f(1)+f(2)}{2}\le M$$

$\exists x_0\in[1,2]$，使 $\frac{f(1)+f(2)}{2}=f(x_0)$

$\Rightarrow f(1)+f(2)=2f(x_0)$

2° $F(x)=\int_2^x f(t)\,dt \quad F'(x)=f(x)$

$\int_2^4 f(x)\,dx=F(4)-F(2)=F'(c)(4-2)=2f(c) \quad (2<c<4)$

3° $f(0)=f(x_0)=f(c)$

3. $f(x)$ 三阶可导，$f(1)=0$，$F(x)=x^3f(x)$

证 $\exists\xi\in(0,1)$，使 $F'''(\xi)=0$

证：$F(0)=F(1)=0$

$\exists\xi_1\in(0,1)$，使 $F'(\xi_1)=0$，$F'(x)=3x^2f(x)+x^3f'(x)$. $F'(0)=0$

$F'(0)=F'(\xi_1)=0$.

$\exists\xi_2\in(0,\xi_1)$ 使 $F''(\xi_2)=0$

$F''(x)=6xf(x)+6x^2f'(x)+x^3f''(x) \qquad F''(0)=0$.

$F''(0)=F''(\xi_2)=0$.

$\exists\xi\in(0,\xi_2)\subset(0,1)$，$F'''(\xi)=0$

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题型三 仅有ξ

① 还原法 （两[项]，导数差一阶）

1. $f(x)\in C[0,1]$，$(0,1)$内可导. $f(1)=3\int_0^{\frac{1}{3}}e^{1-x}f(x)dx$

证 $\exists\xi\in(0,1)$，使 $f'(\xi)=f(\xi)$

证：令 $\varphi(x)=e^{-x}f(x)$

$f(1)=3\int_0^{\frac{1}{3}}e^{1-x}f(x)dx=3\times e^{1-c}f(c)\times\frac{1}{3}=e^{1-c}f(c)\quad(0\le c\le\frac{1}{3})$

$\Rightarrow\ e^{-1}f(1)=e^{-c}f(c)\ \Rightarrow\ \varphi(c)=\varphi(1)$

$\exists\xi\in(c,1)\subset(0,1)$，使 $\varphi'(\xi)=0$.

而 $\varphi'(x)=e^{-x}[f'(x)-f(x)]$ 且 $e^{-x}\ne0$.

$\therefore\ f'(\xi)=f(\xi)$

2. $f(x)\in C[0,1]$，$(0,1)$内二阶可导，$f(0)=f(1)$

证 $\exists\xi\in(0,1)$，$f''(\xi)=\frac{2f'(\xi)}{1-\xi}$

证：令 $\varphi(x)=(x-1)^2f'(x)$　　$\varphi(1)=0$

$f(0)=f(1)\ \Rightarrow\ \exists c\in(0,1)$，使 $f'(c)=0\Rightarrow\varphi(c)=0$.

$\varphi(c)=\varphi(1)=0$

$\exists\xi\in(c,1)\subset(0,1)$，使 $\varphi'(\xi)=0$.

而 $\varphi'(x)=2(x-1)f'(x)+(x-1)^2f''(x)$

$\therefore\ 2(\xi-1)f'(\xi)+(\xi-1)^2f''(\xi)=0$

$\because\ \xi-1\ne0\quad\therefore\ \cancel{2f'(\xi)}\ \frac{2f'(\xi)}{\xi-1}+f''(\xi)=0.$

② 分组法.

1. $f(x)\in C[0,1]$，$(0,1)$内可导. $f(0)=0$，$f(\frac{1}{2})=1$，$f(1)=\frac{1}{2}$

证 1) $\exists c\in(0,1)$，$f(c)=c$

2) $\exists\xi\in(0,1)$，$f'(\xi)+2f(\xi)=1+2\xi$

证：1) $h(x)=f(x)-x$.　$h(\frac{1}{2})=\frac{1}{2}>0$　　$h(1)=-\frac{1}{2}<0$.

$\Rightarrow\exists c\in(\frac{1}{2},1)\subset(0,1)$，$h(c)=0\ \Rightarrow\ f(c)=c$

2) $\varphi(x)=e^{2x}[f(x)-x]$　　$\varphi(c)=0$，$\varphi(0)=0$

$\exists\xi\in(0,c)\subset(0,1)$，使 $\varphi'(\xi)=0$.

而 $\varphi'(x)=e^{2x}[f'(x)-1+2f(x)-2x]$ 且 $e^{2x}\ne0$

$\therefore\ f'(\xi)-1+2f(\xi)-2\xi=0$

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2. $f(x)$ 二阶可导，$\lim\limits_{x\to 0}\frac{f(x)}{x}=1$，$f(1)=1$

证 $\exists\xi\in(0,1)$，$f''(\xi)-f'(\xi)+1=0$

证：令 $\varphi(x)=e^{-x}[f'(x)-1]$

$\lim\limits_{x\to 0}\frac{f(x)}{x}=1\Rightarrow f(0)=0$，$f'(0)=1$

$\exists c\in(0,1)$，使 $f'(c)=\frac{f(1)-f(0)}{1-0}=1$

$\Rightarrow\varphi(0)=\varphi(c)=0$

3. $f(x)\in C[a,b]$，(a,b) 内可导，$f(a)=a$，$\int_a^b f(x)dx=\frac{1}{2}(b^2-a^2)$

证 $\exists\xi\in(0,1)$，使 $f'(\xi)-f(\xi)+\xi=1$

证 令 $\varphi(x)=e^{-x}[f(x)-x]$，$\varphi(a)=0$

$\int_a^b f(x)dx=\int_a^b x\,dx\Rightarrow\int_a^b[f(x)-x]dx=0$

令 $F(x)=\int_a^x[f(t)-t]dt$　$F'(x)=f(x)-x$

$\because F(a)=F(b)=0$

$\exists c\in(a,b)$ 使 $F'(c)=0\Rightarrow f(c)-c=0$.

$\varphi(c)=0$

4. $f(x)\in C[a,b]$，(a,b) 内二阶可导，$f(a)=f(b)=\int_a^b f(x)dx=0$

证① $\exists\xi_1,\xi_2\in(a,b)$，使 $\begin{cases}f'(\xi_1)+f(\xi_1)=0\\ f'(\xi_2)+f(\xi_2)=0\end{cases}$

② $\exists\xi\in(a,b)$，使 $f''(\xi)=f(\xi)$

证：① 令 $F(x)=\int_a^x f(t)dt$　$F'(x)=f(x)$

$F(a)=F(b)=0$

$\exists c\in(a,b)$，$F'(c)=0\Rightarrow f(c)=0$

令 $h(x)=e^x f(x)$

$\because h(a)=h(c)=h(b)=0$

$\therefore\exists\xi_1\in(a,c)$，$\xi_2\in(c,b)$，使 $h'(\xi_1)=0$，$h'(\xi_2)=0$.

而 $h'(x)=e^x\cdot[f'(x)+f(x)]$ 且 $e^x\neq 0$

$\therefore\begin{cases}f'(\xi_1)+f(\xi_1)=0\\ f'(\xi_2)+f(\xi_2)=0\end{cases}$

$f''-f=0 \Rightarrow f''+f'-f'-f=0 \qquad (f'+f)'-(f'+f)$

Date. /

② $\varphi(x)=e^{-x}[f'(x)+f(x)]$

$\because \varphi(\xi_1)=\varphi(\xi_2)=0$

$\therefore \exists\xi\in(\xi_1,\xi_2)\subset(a,b)$，使 $\varphi'(\xi)=0$.

而 $\varphi'(x)=e^{-x}[f''(x)-f(x)]$ 且 $e^{-x}\neq 0$.

③ 凑微法

$\left\{\begin{array}{l}\xi\to x\\ 去分母移项\end{array}\right. \Rightarrow 和+=0 \Rightarrow (\underline{\quad ? \quad})'=0$

5. $f(x), g(x)\in C[a,b]$，(a,b)内二阶可导，$g(x)g''(x)\neq 0 \quad (a<x<b)$.

$f(a)=g(a)=f(b)=g(b)=0$.

证：$\exists\xi\in(a,b)$，使 $\frac{f''(\xi)}{g''(\xi)}=\frac{f(\xi)}{g(\xi)}$

分析：$f''(x)g(x)-f(x)g''(x)=0 \Rightarrow [f'(x)g(x)-f(x)g'(x)]'=0$

证：令 $\varphi(x)=f'(x)g(x)-f(x)g'(x)$

$\varphi(a)=\varphi(b)=0$

型 四 有a, b, ξ.

① ξ与a, b可分离.

ξ与a, b分离 $\Rightarrow$ a, b侧 $\left\{\begin{array}{ll}\frac{f(b)-f(a)}{b-a} & - L\\ \frac{f(b)-f(a)}{g(b)-g(a)} & - C\end{array}\right.$

1. $f(x)$在$[a,b]$上二阶可导，$\lim\limits_{x\to a^+}\frac{f(x)}{x-a}=0$，$f''(x)>0 \quad (a<x<b)$

证 1) $f(x)>0 \quad (a<x\leq b)$； 2) $\exists\xi,\eta\in(a,b)$，使 $\frac{b^2-a^2}{\int_a^b f(x)dx}=\frac{2\xi}{f(\xi)}=\frac{2\xi}{f'(\eta)(\xi-a)}$

证：① $\lim\limits_{x\to a^+}\frac{f(x)}{x-a}=0 \Rightarrow f(a)=0, f'(a)=0$. 一致/推荐

$\left\{\begin{array}{l}f'(a)=0\\ f''(x)>0 \quad (a<x<b)\end{array}\right. \Rightarrow f'(x)>0 \quad (a<x<b)$

$\left\{\begin{array}{l}f(a)=0\\ f'(x)>0 \quad (a<x<b)\end{array}\right. \Rightarrow f(x)>0 \quad (a<x\leq b)$

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③ $F(x)=\int_a^x f(t)dt$ $F'(x)=f(x)>0$ $(a<x<b)$.

左 $=\frac{b^2-a^2}{F(b)-F(a)}=\frac{2\xi}{F'(\xi)}=\frac{2\xi}{f(\xi)}$，$\xi\in(a,b)$

$f(\xi)=f(\xi)-f(a)=f'(\eta)(\xi-a)$ $(a<\eta<\xi)$

② ξ与 a,b 不可分 — 凑微法

$\begin{cases}\xi\to x\\ 去分母移项\end{cases}$ $\Rightarrow$ 式子 $=0$ $\Rightarrow$ (?)$'=0$

1. $f(x).g(x)\in C[a,b]$ 证 $\exists\xi\in(a,b)$，使 $f(\xi)\int_a^\xi g(t)dt=g(\xi)\int_\xi^b f(t)dt$

分析：$f(x)\int_a^x g(t)dt+g(x)\int_b^x f(t)dt=0$

$[\int_b^x f(t)dt\int_a^x g(t)dt]'=0$

证：令 $\varphi(x)=\int_b^x f(t)dt\int_a^x g(t)dt$

$\varphi(a)=\varphi(b)=0$

2. $f(x)$，$g(x)\in C[a,b]$，(a,b)内可导，$g'(x)\neq0$ $(a<x<b)$.

证 $\exists\xi\in(a,b)$，使 $\frac{f(\xi)-f(a)}{g(b)-g(\xi)}=\frac{f'(\xi)}{g'(\xi)}$

分析：$f(x)g'(x)-f(a)g'(x)-f'(x)g(b)+f'(x)g(x)=0$

$[f(x)g(x)-f(a)g(x)-f(x)g(b)]'=0$

证：令 $\varphi(x)=f(x)g(x)-f(a)g(x)-f(x)g(b)$

$\varphi(a)=-f(a)g(b)=\varphi(b)$

型五的中值

① $f(\xi)$，$f'(\eta)$. —— $\begin{cases}找三点\\ 2L\end{cases}$

1. $f(x)\in C[a,b]$，(a,b)内二阶可导，$f'_+(a)>0$

$f(a)=f(b)$.

证 1) $\exists\xi,\eta\in(a,b)$，$f'(\xi)>0$，$f'(\eta)<0$.

2) $\exists\xi\in(a,b)$，$f''(\xi)<0$.

证：1) $f'_+(a)>0$ $\Rightarrow$ $\exists c\in(a,b)$ 使 $f(c)>f(a)$

($f'_+(a)=\lim\limits_{x\to a^+}\frac{f(x)-f(a)}{x-a}>0$ $\exists\delta>0$，当 $x\in(a,a+\delta)$ 时

$\frac{f(x)-f(a)}{x-a}>0$ $\Rightarrow f(x)>f(a)$)

Date. /

$\exists\, \xi \in (a,c)$, $\eta \in (c,b)$，使

$$f'(\xi) = \frac{f(c)-f(a)}{c-a} > 0,\quad f'(\eta) = \frac{f(b)-f(c)}{b-c} < 0$$

2、$f(x) \in C[0,1]$，$(0,1)$内可导，$f(0)=0$，$f(1)=1$

证 1) $\exists\, c \in (0,1)$，$f(c) = \frac{2}{3}$；

2) $\exists\, \xi, \eta \in (0,1)$，

$$\frac{2}{f'(\xi)} + \frac{1}{f'(\eta)} = 3$$

证：1) $\varphi(x) = f(x) - \frac{2}{3}$　　$\varphi(0) = -\frac{2}{3} < 0$，$\varphi(1) = \frac{1}{3} > 0$

$\exists\, c \in (0,1)$，使 $\varphi(c) = 0 \Rightarrow f(c) = \frac{2}{3}$

2) $\exists\, \xi \in (0,c)$，$\eta \in (c,1)$，使 $f'(\xi) = \frac{f(c)-f(0)}{c} = \frac{2}{3c}$

$$f'(\eta) = \frac{f(1)-f(c)}{1-c} = \frac{1}{3(1-c)}$$

$$\frac{2}{f'(\xi)} = 3c \qquad \frac{1}{f'(\eta)} = 3(1-c)$$

② ξ, η 复杂程度不同

留复杂 $\Rightarrow \begin{cases} (\quad)' & — \ L \\ \frac{(\quad)'}{(\quad)'} & — \ C \end{cases}$

1、P_{58} 例1. 解.

分析：$e^{\eta}[f'(\eta)+f(\eta)] = e^{\xi}$

令 $\varphi(x) = e^{x} f(x)$

$\exists\, \eta \in (a,b)$，使 $\frac{\varphi(b)-\varphi(a)}{b-a} = \varphi'(\eta)$

$$\Rightarrow \frac{e^b - e^a}{b-a} = e^{\eta}\cdot[f'(\eta)+f(\eta)]$$

$\exists\, \xi \in (a,b)$ 使 $\frac{e^b-e^a}{b-a} = e^{\xi}$

2、P_{59} 例3.

分析：　$f'(\xi)$　　$e^{-\eta} f'(\eta) = \frac{f'(\eta)}{e^{\eta}}$

证：令 $g(x) = e^{x}$　　$g'(x) = e^{x} \neq 0$

$\exists\, \eta \in (a,b)$，使 $\frac{f(b)-f(a)}{e^b - e^a} = \frac{f'(\eta)}{e^{\eta}}$

$$\Rightarrow \frac{f(b)-f(a)}{b-a} = \frac{(e^b - e^a)}{b-a} \frac{f'(\eta)}{e^{\eta}}$$

$\exists\, \xi \in (a,b)$ 使 $f'(\xi) = \frac{f(b)-f(a)}{b-a}$

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3. $f(x)\in C[a,b]$，(a,b)内可导 $(a>0)$

证 $\exists \xi,\eta\in(a,b)$，使 $ab f'(\xi)=\eta^2 f'(\eta)$

分析：$\eta^2 f'(\eta)=\dfrac{f'(\eta)}{\frac{1}{\eta^2}}$ $-f(x)$ $--\frac{1}{x}$

证：令 $g(x)=-\frac{1}{x}$，$g'(x)=\frac{1}{x^2}\neq 0$　$a<x<b$ $(\cancel{\quad})$

$\exists \eta\in(a,b)$，$\dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(\eta)}{g'(\eta)}$

$\Rightarrow \dfrac{f(b)-f(a)}{\frac{1}{a}-\frac{1}{b}}=\dfrac{f'(\eta)}{\frac{1}{\eta^2}}$

$\Rightarrow ab\dfrac{f(b)-f(a)}{b-a}=\eta^2 f'(\eta)$

$\exists \xi\in(a,b)$，$f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$

③ ξ,η 皆复杂且对等

如：$f'(\xi)-2\xi=f'(\eta)-2\eta$

一个辅助函数，2L.

$\begin{cases}\varphi(x)=f(x)-x^2\\ \text{找三点，2L}\end{cases}$

$f'(\xi)+\xi=f'(\eta)-2\eta$.

二个辅 $\begin{cases}\varphi_1(x)=f(x)+\frac{1}{2}x^2,\quad \varphi_2(x)=f(x)-x^2\\ \text{找三点，2L.}\end{cases}$

1. $f(x)\in[0,1]$，$f(x)$在$(0,1)$内可导，$f(0)=0$　$f(1)=\frac{1}{3}$.

证 $\exists \xi\in(0,\frac{1}{2})$，$\eta\in(\frac{1}{2},1)$ 使 $f'(\xi)-\xi^2+f'(\eta)-\eta^2=0$

证：$\varphi(x)=f(x)-\frac{1}{3}x^3$

$\exists \xi\in(0,\frac{1}{2})$，$\eta\in(\frac{1}{2},1)$

$\varphi'(\xi)=\dfrac{\varphi(\frac{1}{2})-\varphi(0)}{\frac{1}{2}}=2\varphi(\frac{1}{2})$

$\varphi'(\eta)=\dfrac{\varphi(1)-\varphi(\frac{1}{2})}{1-\frac{1}{2}}=-2\varphi(\frac{1}{2})$

$\varphi'(\xi)+\varphi'(\eta)=0$

而 $\varphi'(x)=f'(x)-x^2$

$\therefore f'(\xi)-\xi^2+f'(\eta)-\eta^2=0$

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題六 L常见用法

① $f(b)-f(a)$、$\frac{f(b)-f(a)}{b-a}$、$f(a)\neq f(b)$ —— L

② $f(a)$、$f(c)$、$f(b)$ —— 2L

③ $f \Rightarrow f'$ $\begin{cases} f(x)-f(a)=f'(\xi)(x-a) \\ f(x)-f(a)=\int_a^x f'(t)dt \end{cases}$

1、P_{61} 例1. 解:

$f(x)-f(x-1)=f'(\xi)$ 　$(x-1<\xi<x)$

左 $=\lim\limits_{x\to\infty} f'(\xi)=e$

右 $=\lim\limits_{x\to\infty}\left[\left(1+\frac{2c}{x-c}\right)^{\frac{x-c}{2c}}\right]^{x\cdot\frac{2c}{x-c}}=e^{2c}$

$\Rightarrow e^{2c}=e$ 　$\Rightarrow c=\frac{1}{2}$

2、$0<a<b$，证：$\frac{\ln b-\ln a}{b-a}>\frac{2a}{a^2+b^2}$

证：令 $f(x)=\ln x$ 　$f'(x)=\frac{1}{x}$ ~~$\neq 0$~~

左 $=f'(\xi)=\frac{1}{\xi}$ 　$(a<\xi<b)$

$\frac{1}{\xi}>\frac{1}{b}>\frac{2a}{a^2+b^2}$

3、$f(x)\in C[a,b]$，$[a,b]$上可导．$|f'(x)|\leq M$.

$f(x)$在(a,b)内至少一个零点．证：

$|f(a)|+|f(b)|\leq M(b-a)$

证：$\exists c\in(a,b)$，$f(c)=0$

$\begin{cases} f(c)-f(a)=f'(\xi)(c-a) & (a<\xi<c) \\ f(b)-f(c)=f'(\eta)(b-c) & (c<\eta<b) \end{cases}$

$\Rightarrow \begin{cases} |f(a)|\leq M(c-a) \\ |f(b)|\leq M(b-c) \end{cases}$

4. $f(x)$在$[0,a]$上二阶可导．$|f''(x)|\leq M$

$f(x)$在$(0,a)$内取最小．

证：$|f'(0)|+|f'(a)|\leq Ma$.

证．$\exists c\in(0,a)$，使 $f(c)=m$．$f'(c)=0$

$\begin{cases} f'(c)-f'(0)=f''(\xi)c & (0<\xi<c) \\ f'(a)-f'(c)=f''(\eta)(a-c) & (c<\eta<a) \end{cases} \Rightarrow \begin{cases} |f'(0)|\leq Mc \\ |f'(a)|\leq M(a-c) \end{cases}$

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5. $f''(x)>0$, $f(0)=0$ 证：$2f(1)<f(2)$.

证：$f(1)-f(0)=f'(\xi_1)$ $(0<\xi_1<1)$

$f(2)-f(1)=f'(\xi_2)$ $(1<\xi_2<2)$

∵ $f''(x)>0$ ，∴ $f'(x)\uparrow$

∵ $\xi_1<\xi_2$ ，∴ $f'(\xi_1)<f'(\xi_2)$

$\Rightarrow$ $f(1)-f(0)<f(2)-f(1)$

型七 $f''(x)>0$ (<0) 二阶导符号

① f''恒$>0 \Rightarrow f'\uparrow$

如：$f(a)=g(a)$ ，$f'(a)=g'(a)$ ，$f''(x)>g''(x)$ $(x>a)$

证：$x>a$时，$f(x)>g(x)$.

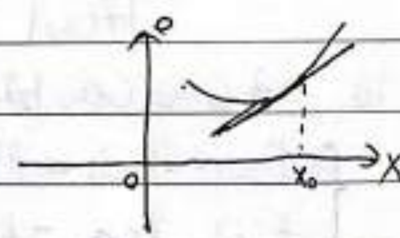
证：令 $\varphi(x)=f(x)-g(x)$.

$\varphi(a)=0$ ，$\varphi'(a)=0$ ，$\varphi''(x)>0$ $(x>a)$

$$\begin{cases}\varphi'(a)=0\\ \varphi''(x)>0\ (x>a)\end{cases} \Rightarrow \varphi'(x)>0 \quad (x>a)$$

$$\begin{cases}\varphi(a)=0\\ \varphi'(x)>0\ (x>a)\end{cases} \Rightarrow \varphi(x)>0 \quad (x>a)$$

② $f''(x)>0$



切线 $y-f(x_0)=f'(x_0)(x-x_0)$

即 $y=f(x_0)+f'(x_0)(x-x_0)$.

$f(x)\geq f(x_0)+f'(x_0)(x-x_0)$

"=" 成立 $\Leftrightarrow$ " $x=x_0$ "

证：$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2!}(x-x_0)^2$

∵ $f''(x)>0$

∴ $f(x)\geq f(x_0)+f'(x_0)(x-x_0)$

"=" $\Leftrightarrow$ " $x=x_0$ "

1、证明 $x\neq 0$ 时 $e^x>1+x$

证：法一 $f(x)=e^x-1-x$，$f(0)=0$

$f'(x)=e^x-1$，$f'(0)=0$

$f''(x)=e^x>0$.

$$\begin{cases} f'(0)=0 \\ f''(x)>0 \end{cases} \Rightarrow \begin{cases} f'(x)<0, & x<0 \\ f'(x)>0, & x>0 \end{cases}$$

$\Rightarrow$ $x=0$ 为 $f(x)$ 最小点，$m=f(0)=0$.

$\therefore$ $x\neq 0$ 时 $f(x)>0$

法二 $f(x)=e^x$ $f''(x)=e^x>0$

则 $x\neq 0$ 时

$f(x)>f(0)+f'(0)x$

即 $e^x>1+x$

2、$f''(x)>0$，$\lim\limits_{x\to 0}\frac{f(x)}{x}=1$

证：$f(x)\geq x$.

证：$\lim\limits_{x\to 0}\frac{f(x)}{x}=1 \Rightarrow f(0)=0$ $f'(0)=1$

法一. $f(x)=f(x)-f(0)=f'(\xi)x$ （ξ在0与x之间）

① $x<0$，$x<\xi<0$

$\because$ $f''(x)>0$ $\therefore$ $f'(x)\uparrow$

$\Rightarrow f'(\xi)<f'(0)=1 \Rightarrow f'(\xi)x>x$

即 $f(x)>x$；

② $x>0$，$0<\xi<x$.

$\Rightarrow f'(0)<f'(\xi) \Rightarrow f'(\xi)>1 \Rightarrow f'(\xi)x>x$ 即 $f(x)>x$

$x=0$ 时 等号成立.

$\therefore$ $f(x)\geq x$

法二：$f''(x)>0$

$\Rightarrow f(x)\geq f(0)+f'(0)x=x$

题八 Taylor

$f(x)=P_n(x)+R_n(x)$

① x_0 $\begin{cases} x_0 — 一阶导数的点 \\ x_0=\frac{a+b}{2} \\ \cdots \end{cases}$

② x $\begin{cases} 函数值对应的点(无导数) \\ 端点 \\ \forall x \\ \cdots \end{cases}$

$f^{(n)}(\xi)$ $(n\geq 2)$

$\begin{cases} f(a)、f(c)、f(b) — 2L \\ f'(a),\ f'(c),\ f'(b) \end{cases}$

$f(a),\ f'(c),\ f(b) — 2T$

$\cancel{f^{(n)}(\xi)\ (n\geq 2)}$

1、$f'''(x)\in C[-1,1]$，$f(-1)=0$，$f'(0)=0$，$f(1)=1$.

证 $\exists\xi\in(-1,1)$，$f'''(\xi)=3$.

证：1° $f(-1)=f(0)+\frac{f''(0)}{2!}(-1-0)^2+\frac{f'''(\xi_1)}{3!}(-1-0)^3$. $(-1<\xi_1<0)$.

$f(1)=f(0)+\frac{f''(0)}{2!}(1-0)^2+\frac{f'''(\xi_2)}{3!}(1-0)^3$ $(0<\xi_2<1)$

$\Rightarrow\begin{cases} 0=f(0)+\frac{1}{2}f''(0)-\frac{1}{6}f'''(\xi_1) \\ 1=f(0)+\frac{1}{2}f''(0)+\frac{1}{6}f'''(\xi_2) \end{cases}$

2° $f'''(\xi_1)+f'''(\xi_2)=6$

3° $f'''(x)\in C[\xi_1,\xi_2]$ $\Rightarrow\exists\, m, M$

$2m\leq 6\leq 2M$ $\Rightarrow$ $m\leq 3\leq M$.

$\exists\xi\in[\xi_1,\xi_2]\subset(-1,1)$，使 $f'''(\xi)=3$.

2. P13, 例2 $f(x)\in C[0,1]$

设：$f(0)=f(1)=0$

$\min\limits_{0\leq x\leq 1}f(x)=-1$ $\Rightarrow\exists c\in(0,1)$，$f(c)=-1$，$f'(c)=0$.

$\begin{cases} f(0)=f(c)+\frac{f''(\xi_1)}{2!}(0-c)^2. & (0<\xi_1<c) \\ f(1)=f(c)+\frac{f''(\xi_2)}{2!}(1-c)^2. & (c<\xi_2<1). \end{cases}$

$\Rightarrow\begin{cases} 0=-1+\frac{c^2}{2}f''(\xi_1) \\ 0=-1+\frac{(1-c)^2}{2}f''(\xi_2) \end{cases}$ $\Rightarrow f''(\xi_1)=\frac{2}{c^2}$，$f''(\xi_2)=\frac{2}{(1-c)^2}$

① $c\in(0,\frac{1}{2}]$时. $f''(\xi_1)\geq 8$.

② $c\in(\frac{1}{2},1)$时. $f''(\xi_2)\geq 8$.

二、单调性与极值

$y=f(x)$

1° $x\in D$；

2° $f'(x)\begin{cases}=0 & (不一定)；\\ 无 & \end{cases}$

3° 判别法

方法一：

① $\begin{cases}x<x_0, & f'<0\\ x>x_0, & f'>0\end{cases} \Rightarrow x_0$ 极小点；

② $\begin{cases}x<x_0, & f'>0\\ x>x_0, & f'<0\end{cases} \Rightarrow x_0$ 极大点；

方法二：

$f'(x_0)=0 \quad f''(x_0)\begin{cases}>0 & 小，\\ <0 & 大，\end{cases}$

补充：

Q1. $f'(a)=f''(a)=0$，$f'''(a)>0\ (<0)$，$x=a$?

$f'''(a)=\lim\limits_{x\to a}\dfrac{f''(x)}{x-a}>0$

$\exists\,\delta>0$ 当 $0<|x-a|<\delta$ 时，$\dfrac{f''(x)}{x-a}>0$.

$\begin{cases}x\in(a-\delta,a). & f''(x)<0\\ x\in(a,a+\delta) & f''(x)>0\end{cases}$

$\Rightarrow (a,f(a))$ 拐点，

① $f'(a)=\cdots=f^{(2k)}(a)=0 \quad f^{(2k+1)}(a)\neq 0$

$\Rightarrow (a,f(a))$ 为拐点，

Q2. $f'(a)=f''(a)=f'''(a)=0.$ $f^{(4)}(a)>0\ (<0)$，$x=a$?

$f^{(4)}(a)=\lim\limits_{x\to a}\dfrac{f'''(x)}{x-a}>0$

$\exists\,\delta>0$，当 $0<|x-a|<\delta$ 时，$\dfrac{f'''(x)}{x-a}>0$

$\begin{cases}x\in(a-\delta,a). & f'''(x)<0.\\ x\in(a,a+\delta), & f'''(x)>0.\end{cases} \Rightarrow x=a$ 为 $f''(x)$ 极小点，而 $f''(a)=0$.

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$$\begin{cases} x\in(a-\delta,a), & f''>0 \\ x\in(a,a+\delta), & f''>0 \end{cases} \Rightarrow f'(x) 在 (a-\delta,a+\delta)\uparrow$$

$\because f'(a)=0 \quad \therefore \begin{cases} x\in(a-\delta,a), & f'<0 \\ x\in(a,a+\delta), & f'>0 \end{cases}$

$x=a$ 为极小值

② $f'(a)=\cdots=f^{(2k-1)}(a)=0.\quad f^{(2k)}(a)\begin{cases} >0 & 小 \\ <0 & 大. \end{cases}$

三 导数应用.

(一) 凹凸性.

1. def — ① If $\forall x_1, x_2 \in D$ 且 $x_1\neq x_2$.

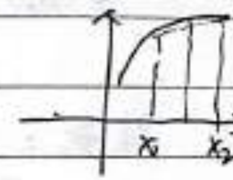
$f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}$

凹

② If $\forall x_1, x_2 \in D$ 且 $x_1\neq x_2$.

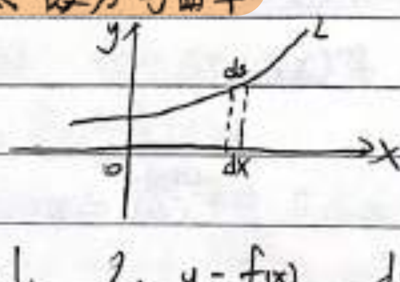
$f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}$

凸



2. 判别法.

(二) 弧微分与曲率



$\Delta s,\ \Delta y,\ \Delta x$: $(\Delta s)^2 \approx (\Delta x)^2+(\Delta y)^2$

$ds,\ dy,\ dx$: $ds=\sqrt{(dx)^2+(dy)^2}$

Case 1. L: $y=f(x)$ $\quad ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+f'^2(x)}\,dx$

Case 2. L $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$ $\quad ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt.$

Case 3. L: $r(\theta)$

$\Rightarrow \begin{cases} x=r(\theta)\cos\theta \\ y=r(\theta)\sin\theta \end{cases}$ $\quad ds=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$

$=\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$

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曲率：$K=\frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}$　　$y'=\frac{dy}{dx}$　$y''=\frac{d^2y}{dx^2}$

$R=\frac{1}{K}$

(三) 渐近线.

例1. $L: y=f(x)=\int_0^x e^{-t^2}dt$，求水平渐近线.

解：$\lim\limits_{x\to+\infty} y=\int_0^{+\infty} e^{-t^2}dt \overset{令t^2=u}{=\!=} \int_0^{+\infty} e^{-u}\cdot\frac{1}{2\sqrt{u}}du=\frac{1}{2}\int_0^{+\infty}u^{-\frac{1}{2}}e^{-u}du$

$=\frac{1}{2}\Gamma(\frac{1}{2})=\frac{\sqrt{\pi}}{2}$

$y=\frac{\sqrt{\pi}}{2}$ 为水平渐近线

$\lim\limits_{x\to-\infty} y=\int_0^{-\infty} e^{-t^2}dt \overset{t=-u}{=\!=} \int_0^{+\infty} e^{-u^2}(-du)=-\int_0^{+\infty} e^{-t^2}dt=-\frac{\sqrt{\pi}}{2}$

$y=-\frac{\sqrt{\pi}}{2}$ 为水平渐近线.

例2. $y=\frac{2x+1}{x}e^{\frac{1}{x-1}}$　求渐近线

解：$\lim\limits_{x\to\infty} y=2$　　$y=2$ 为水平渐近线.

$\lim\limits_{x\to0} y=\infty$　　$x=0$ 为铅直渐近线.

$y(1-0)=0$.　$y(1+0)=+\infty$　　$x=1$ 为铅直渐近线.

3. $y=\sqrt{x^2-4x+8}+x$

解：$\lim\limits_{x\to-\infty} y=\lim\limits_{x\to-\infty}\frac{-4x+8}{\sqrt{x^2-4x+8}-x}=2$

$y=2$ 为水平渐近线

$\lim\limits_{x\to+\infty}\frac{y}{x}=2$　　$\lim\limits_{x\to+\infty}(y-2x)=\lim\limits_{x\to+\infty}(\sqrt{x^2-4x+8}-x)=\lim\limits_{x\to+\infty}\frac{-4x+8}{\sqrt{x^2-4x+8}+x}=-2$

$y=2x-2$

4. $y=(2x-1)e^{\frac{1}{x}}$ 斜一

解 $\lim\limits_{x\to\infty}\frac{y}{x}=2$

$\lim\limits_{x\to\infty}(y-2x)=\lim\limits_{x\to\infty}[(2x-1)e^{\frac{1}{x}}-2x]=\lim\limits_{x\to\infty}[2x(e^{\frac{1}{x}}-1)-e^{\frac{1}{x}}]$

$=2\lim\limits_{x\to\infty}\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}-1=1$.

$y=2x+1$

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题一 极值点判断

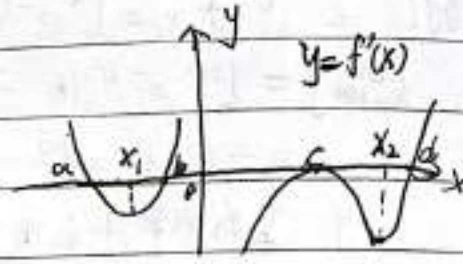
$y=f(x)$:

1° $x\in D$

2° $f'(x)\begin{cases}=0\\ 无\end{cases}$

3° 判断

1. $f(x)\in C[-\infty,+\infty)$.



① 极值点 ② 拐点

解: ① $f'(x)\begin{cases}=0\\ 无\end{cases}\Rightarrow x=a,b,0,c,d$

$\begin{cases}x<a, & f'(x)>0\\ x>a, & f'<0\end{cases}$ $x=a$ 大 $x=b$ 小 $x=0$ 大

$x=c$ ✗ $x=d$ 小

② $f''(x)\begin{cases}=0\\ 无\end{cases}\Rightarrow x=x_1, 0, c, x_2$

✗ ✓ ✓

$\begin{cases}x<x_1 & f''<0\\ x>x_1 & f''>0\end{cases}$ $(x_1,f(x_1))$ 拐点

2. $f'(1)=0$ $\lim\limits_{x\to1}\dfrac{f'(x)}{\sin^2 x}=2$, $x=1$?

解: $\exists\,\delta>0$, 当 $0<|x-1|<\delta$ 时 $\dfrac{f'(x)}{\sin^2 x}>0$

$x\in(1-\delta,1)$ 时, $f'(x)>0$; $\Rightarrow x=1$ 为 极大点

$x\in(1,1+\delta)$ 时, $f'(x)<0$.

3. $f(x)$ 连续可导. $f(0)=0$, $\lim\limits_{x\to0}\dfrac{f(x)+f'(x)}{x}=2$ $x=0$?

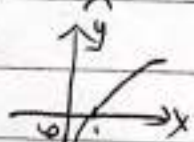
解: $\lim\limits_{x\to0}\dfrac{f(x)+f'(x)}{x}=2\Rightarrow f(0)+f'(0)=0$

$\because f(0)=0$ $\therefore f'(0)=0$

$2=\lim\limits_{x\to0}\dfrac{f(x)-f(0)}{x}+\lim\limits_{x\to0}\dfrac{f'(x)-f'(0)}{x}=f'(0)+f''(0)=f''(0)>0$

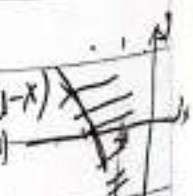
4. $y=|\ln(1-x)|$

移 $(0,0)$ 解: $y=\ln x$

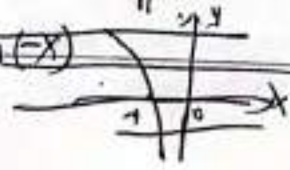


$y=\ln(x-1)$

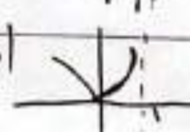
$y=\ln(1-x)$
$=\ln[-(x-1)]$

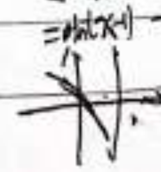


$y=\ln(-x)$



$y=|\ln(1-x)|$





题二. 不等式证明.

~~证~~ ① 单调性.

② 中值定理 $\begin{cases} \frac{f(b)-f(a)}{b-a}、\frac{f(b)-f(a)}{g(b)-g(a)} \\ f、f' \text{在同一式中} \end{cases}$

③ 凹凸法

$$\text{If} \begin{cases} f(a)=f(b)=0 \\ f''(x)<0 \quad a<x<b \end{cases} \Rightarrow f(x)>0 \quad (a<x<b)$$

例1. 证 当 $0<x<\frac{\pi}{2}$ 时, $\frac{2}{\pi}x<\sin x<x$

证 令 $f(x)=x-\sin x$, $f(0)=0$.

$$f'(x)=1-\cos x>0 \quad (0<x<\frac{\pi}{2})$$

$$\begin{cases} f(0)=0 \\ f'(x)>0 \quad (0<x<\frac{\pi}{2}) \end{cases} \Rightarrow f(x)>0 \quad (0<x<\frac{\pi}{2})$$

令 $g(x)=\sin x-\frac{2}{\pi}x \quad g(0)=0, \quad g(\frac{\pi}{2})=0$

$g'(x)=\cos x-\frac{2}{\pi}$

$g''(x)=-\sin x<0 \quad (0<x<\frac{\pi}{2})$

$\therefore g(x)>0 \quad (0<x<\frac{\pi}{2})$

④ m. M法

If 证 $f(x)\geq m$, $f(x)\leq M$.

P68. 例10. 证

令 $\varphi(x)=1+x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}$, $\varphi(0)=0$.

$$\varphi'(x)=\ln(x+\sqrt{1+x^2})+x\cdot\frac{1}{x+\sqrt{1+x^2}}\cdot(1+\frac{x}{\sqrt{1+x^2}})-\frac{x}{\sqrt{1+x^2}}$$

$$=\ln(x+\sqrt{1+x^2}). \quad \varphi'(0)=0$$

$$\varphi''(x)=\frac{1}{x+\sqrt{1+x^2}}(1+\frac{x}{\sqrt{1+x^2}})=\frac{1}{\sqrt{1+x^2}}>0$$

$$\begin{cases} \varphi'(0)=0 \\ \varphi''(x)>0 \end{cases} \Rightarrow \begin{cases} \varphi'(x)<0, & x<0 \\ \varphi'(x)>0, & x>0 \end{cases}$$

$\Rightarrow x=0$ 为 $\varphi(x)$ 最小值

$\therefore m=\varphi(0)=0$

$\therefore \varphi(x)\geq 0$.

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P68. 例11 证: $\ln\frac{b}{a} > \frac{2(b-a)}{a+b} \Leftrightarrow (a+b)(\ln b-\ln a)-2(b-a)>0$

令 $f(x)=(a+x)(\ln x-\ln a)-2(x-a)$;　$f(a)=0$

$f'(x)=\ln x-\ln a+\frac{a}{x}-1$,　$f'(a)=0$

$f''(x)=\frac{1}{x}-\frac{a}{x^2}=\frac{x-a}{x^2}>0\ (x>a)$.

$\begin{cases} f'(a)=0 \\ f''(x)>0\quad (x>a) \end{cases} \Rightarrow f'(x)>0\quad (x>a)$

$\begin{cases} f(a)=0 \\ f'(x)>0\ (x>a) \end{cases} \Rightarrow f(x)>0\quad (x>a)$

$\because b>a \quad \therefore f(b)>0$

P68. 例12 证 令 $f(x)=(x^2-1)\ln x-(x-1)^2$　$f(1)=0$

$f'(x)=2x\ln x+x-\frac{1}{x}-2(x-1)$

$=2x\ln x-x-\frac{1}{x}+2$.　$f'(1)=0$

$f''(x)=2\ln x+1+\frac{1}{x^2}$,　$f''(1)=2$

$f'''(x)=\frac{2}{x}-\frac{2}{x^3}=\frac{2(x^2-1)}{x^3}\ \begin{cases} <0, & 0<x<1 \\ >0, & x>1 \end{cases}$

$\Rightarrow x=1$ 为 $f''(x)$ 最小点

而 $f''(1)=2$　$\therefore f''(x)>0$

$\begin{cases} f'(1)=0 \\ f''(x)>0 \end{cases} \Rightarrow \begin{cases} f'(x)>0, & 0<x<1 \\ f'(x)<0, & x>1 \end{cases}$

$\Rightarrow x=1$ 为 $f(x)$ 最小点

$\therefore m=f(1)=0$

$\therefore f(x)\geq 0$

P69. 例14.

1° 左 $=\frac{2\ln c}{c}$ $(a<c<b)$

即证 $\frac{2\ln c}{c}>\frac{4}{e^2}$

2° 令 $\varphi(x)=\frac{2\ln x}{x}$　$(e\leq x\leq e^2)$

$\varphi'(x)=2\frac{1-\ln x}{x^2}<0\ (e<x<e^2)$

$\Rightarrow \varphi(x)$在$[e,e^2]\downarrow$

$\therefore m = \varphi(e^2) = \frac{4}{e^2}$

而 $c \in (e, e^2)$

$\therefore \varphi(c) > \frac{4}{e^2}$

型三 方程解(函数零点)

① 零点定理

② Rolle. $\underset{F(x)}{\overset{f(x):}{\Downarrow}}$

If $F(a) = F(b)$, then $\exists c \in (a,b)$, $F'(c)=0 \Rightarrow f(c)=0$.

1. $a_0 + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$.

证: $a_0 + a_1x + \cdots + a_nx^n = 0$ 至少一正根

证: 令 $f(x) = a_0 + a_1x + \cdots + a_nx^n$

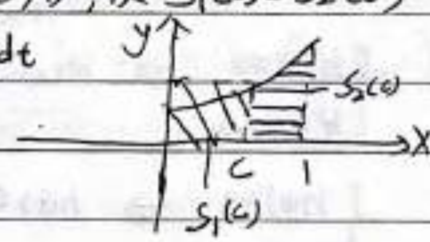
$F(x) = a_0x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}$, $F'(x) = f(x)$

$F(0) = F(1) = 0$.

$\exists c \in (0,1)$, 使 $F'(c)=0 \Rightarrow f(c)=0$

2. $f(x) \in C[0,1]$, $f(x) \geq 0$. 证$\exists c \in (0,1)$, 使 $S_1(c) = S_2(c)$

证: 1° $S_1(c) = cf(c)$ $\quad S_2(c) = \int_c^1 f(t)\,dt$



2° $\varphi(x) = xf(x) - \int_x^1 f(t)\,dt$.

令 $F(x) = x\int_1^x f(t)\,dt$, $F'(x) = \varphi(x)$

$F(0) = F(1) = 0$

$\exists c \in (0,1)$, $F'(c)=0 \Rightarrow \varphi(c)=0 \Rightarrow S_1(c) = S_2(c)$

③ 单调性. $\begin{cases} f(x) \quad (-1<x<1) \\ f'(x), \text{极值} \\ \text{两侧}. \end{cases}$

1. 证明 $\ln x = \frac{x}{e} - \int_0^\pi \sqrt{1-\cos 2x}\,dx$ 有两个正根

证: $\int_0^\pi \sqrt{1-\cos 2x}\,dx = \int_0^\pi \sqrt{2\sin^2 x}\,dx = \sqrt{2}\int_0^\pi \sin x\,dx = 2\sqrt{2}$

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1° $f(x) = \ln x - \frac{x}{e} + 2\sqrt{2} \quad (x>0)$

2° $f'(x) = \frac{1}{x} - \frac{1}{e}$

令 $f'(x)=0 \Rightarrow x=e$

$\because f''(e) = -\frac{1}{e^2} < 0$

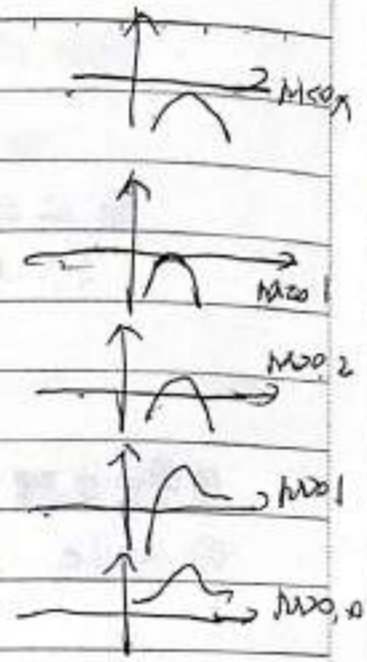
$\therefore$ $x=e$ 为最大点

$M = f(e) = 2\sqrt{2} > 0$.

3° $\lim_{x\to 0^+} f(x) = -\infty$.

$\lim_{x\to +\infty} f(x) = -\infty$

$\therefore$ 方程 2个正根.



2. 方程 $\frac{1}{\ln(1+x)} - \frac{1}{x} = k \ (x>0)$ 有根.

求k范围.

解: 令 $f(x) = \frac{1}{\ln(1+x)} - \frac{1}{x}$

$f'(x) = -\frac{1}{(1+x)\ln^2(1+x)} + \frac{1}{x^2} = \frac{(1+x)\ln^2(1+x) - x^2}{x^2(1+x)\ln^2(1+x)}$

令 $h(x) = (1+x)\ln^2(1+x) - x^2$, $h(0)=0$

$h'(x) = \ln^2(1+x) + 2\ln(1+x) - 2x \qquad h'(0)=0$.

$h''(x) = \frac{2\ln(1+x)}{1+x} + \frac{2}{1+x} - 2$

$= \frac{2[\ln(1+x) - x]}{1+x} < 0 \quad (x>0)$

$\begin{cases} h'(0)=0 \\ h''(x)<0 \end{cases} \Rightarrow h'(x)<0 \quad (x>0)$

$\begin{cases} h(0)=0 \\ h'(x)<0 \end{cases} \Rightarrow h(x)<0 \quad (x>0)$

$\therefore f'(x)<0 \Rightarrow f(x)$在$(0,+\infty)\downarrow$

而 $\lim_{x\to 0^+} f(x) = \frac{1}{2} \qquad \lim_{x\to +\infty} f(x) = 0$

$\therefore \quad 0 < k < \frac{1}{2}$

$y=f(x)$

$F(x,y)=0$

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1° $F(x,y)=0$ 两边对 x 求导　令 $y'=0$ ⇒ $y=y(x)$ 代入 $F(x,y)=0$

$\Rightarrow \begin{cases} x=? & 驻点 \\ y=? \end{cases}$

2° 两边再对 x 求导 ⇒ $y''(?)= \begin{matrix} >0 \\ <0 \end{matrix}$

Part Ⅲ 多元函数微分学

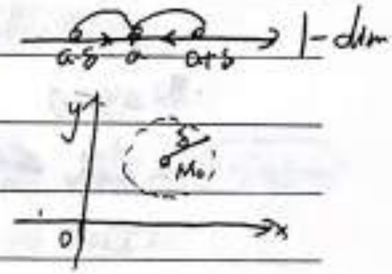
一、defs.

1. 极限 — If $\forall \varepsilon>0$, $\exists \delta>0$.

当 $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$ 时.

$|f(x,y)-A|<\varepsilon$.

$\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A$



2. 连续 — If $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=f(x_0,y_0)$

3. 偏导数 — $f_x'(x_0,y_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=\lim\limits_{x\to x_0}\dfrac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$

4. 可(全)微　$z=f(x,y)$ ($(x,y)\in D$) $M_0(x_0,y_0)\in D$

$\Delta z = f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)\ (= f(x,y)-f(x_0,y_0))$

$\underline{\underline{\text{if}}}\ A\Delta x + B\Delta y + o(\rho)$. $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$

称 $z=f(x,y)$ 在 M_0 可微

$\underset{\|}{A\Delta x + B\Delta y} \triangleq dz|_{M_0}$

$Adx+Bdy$

二、理论.

1. 关系

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证：①"可微 ⇒ 连续"

$\Delta z = f(x,y) - f(x_0,y_0) = A(x-x_0) + B(y-y_0) + o(\sqrt{(x-x_0)^2+(y-y_0)^2})$

$\because \lim\limits_{\substack{x\to x_0\\ y\to y_0}} \Delta z = 0$

$\therefore \lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f(x_0,y_0)$

②"可微 ⇒ 可偏导"

$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$

$= A\Delta x + B\Delta y + o(\rho)$

取 $\Delta y=0$ $\quad \Delta z_x = A\Delta x + o(\Delta x) \Rightarrow \frac{\Delta z_x}{\Delta x} = A + \frac{o(\Delta x)}{\Delta x}$

$\lim\limits_{\Delta x\to 0} \frac{\Delta z_x}{\Delta x} = A \Rightarrow f_x'(x_0,y_0) = A$

同理 $f_y'(x_0,y_0) = B$.

反例1. $z = f(x,y) = \sqrt{x^2+y^2}$ 在 $(0,0)$ 连续.

$f_x'(x_0,y_0) = \lim\limits_{x\to x_0} \frac{f(x,y_0)-f(x_0,y_0)}{x-x_0} = \lim\limits_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0}\frac{|x|}{x}$ 不存在

⇒ $f(x,y)$ 在 $(0,0)$ 对 x 不可偏导.

同理 $f(x,y)$ 在 $(0,0)$ 对 y 不可偏导.

反例2. $z = f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

$\lim\limits_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim\limits_{x\to 0}\frac{0-0}{x} = 0 \Rightarrow f_x'(0,0) = 0$;

$\lim\limits_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = 0 \Rightarrow f_y'(0,0) = 0$.

$\lim\limits_{\substack{x\to 0\\ y=x}} f(x,y) = \frac{1}{2} \neq \lim\limits_{\substack{x\to 0\\ y=-x}} f(x,y) = -\frac{1}{2}$

$\therefore \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ 不存在

$\therefore f(x,y)$ 在 $(0,0)$ 不连续.

Notes:

① $\Delta z = A\Delta x + B\Delta y + o(\rho) \Rightarrow \begin{cases} A = f_x'(x_0,y_0) \\ B = f_y'(x_0,y_0) \end{cases}$

~~② $\Delta z = A\Delta x + B\Delta y + o(\rho)$~~

$\Updownarrow$

~~$\Delta z - A\Delta x - B\Delta y = o(\rho)$~~

② $\Delta z = A\Delta x + B\Delta y + o(\rho)$

$\Updownarrow$

$\Delta z - A\Delta x - B\Delta y = o(\rho)$

$\Updownarrow$

$\lim_{\rho\to 0} \frac{\Delta z - A\Delta x - B\Delta y}{\rho} = 0$

若 $z=f(x,y)$ 在 M_0 可偏导，则

可微 $\Leftrightarrow \lim_{\rho\to 0} \frac{\Delta z - A\Delta x - B\Delta y}{\rho} = 0$

例. $f(x,y)$ 连续 $z=f(x,y) = 2x-3y+7+o(\sqrt{(x-1)^2+(y-2)^2})$

求 $dz|_{(1,2)}$?

解: $f(1,2)=3$，$\Delta x = x-1$，$\Delta y = y-2$，$\rho = \sqrt{(x-1)^2+(y-2)^2}$

$\Delta z = f(x,y) - f(1,2) = f(x,y) - 3 = 2x-3y+4+o(\rho)$

$= 2(x-1) - 3(y-2) + o(\rho)$

$dz|_{(1,2)} = 2dx - 3dy$ $\quad f_x'(1,2)=2,\ f_y'(1,2)=-3$

反例3 $z=f(x,y)=\begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

1° $0 \le |f(x,y)| = |x| \cdot \frac{|y|}{\sqrt{x^2+y^2}} \le |x|$

$\because \lim_{\substack{x\to 0\\ y\to 0}} |x| = 0 \quad \therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 = f(0,0)$

2° $\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = 0 \Rightarrow f_x'(0,0)=0$.

同理 $f_y'(0,0)=0$

3° $\Delta z = f(x,y) - f(0,0) = \frac{xy}{\sqrt{x^2+y^2}}$，$\rho = \sqrt{x^2+y^2}$

$A=0$，$B=0$.

$\lim_{\rho\to 0} \frac{\Delta z - A\Delta x - B\Delta y}{\rho} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$ 不存在

$\therefore f(x,y)$ 在 $(0,0)$ 不可微.

三 求偏导

(一) 显函数

$z = (x^2+y^2)^{e^x \sin 2y}$，求 $\frac{\partial z}{\partial x}$

解: $z = e^{e^x \sin 2y \cdot \ln(x^2+y^2)}$

$\frac{\partial z}{\partial x} = (x^2+y^2)^{e^x \sin 2y} [e^x \sin 2y \cdot \ln(x^2+y^2) + e^x \sin 2y \cdot \frac{2x}{x^2+y^2}]$

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(二) 复合函数求偏导

1. $z=f(e^{2t},\sin^2 t)$. f二阶连续可偏导, 求 $\frac{d^2z}{dt^2}$

解: $\frac{dz}{dt}=2e^{2t}f_1'+\sin 2t f_2'$

$$\frac{d^2z}{dt^2}=4e^{2t}f_1'+2e^{2t}\cdot(2e^{2t}f_{11}''+\sin 2t f_{12}'')+2\cos 2t f_2'+\sin 2t(2e^{2t}f_{21}''+\sin 2t f_{22}'')$$

2. $z=f(x^2+y^2,\frac{y}{x})$, f二阶连续可偏导, 求 $\frac{\partial^2 z}{\partial x\partial y}$

解: $\frac{\partial z}{\partial x}=2xf_1'-\frac{y}{x^2}f_2'$

$$\frac{\partial^2 z}{\partial x\partial y}=2x(2yf_{11}''+\frac{1}{x}f_{12}'')-\frac{1}{x^2}f_2'-\frac{y}{x^2}\cdot(2yf_{21}''+\frac{1}{x}f_{22}'')$$

3. $z=f(x^2+y^2,\frac{y}{x},\sin^2 x)$. f二阶连续可偏导, 求 $\frac{\partial^2 z}{\partial x\partial y}$

解: $\frac{\partial z}{\partial x}=2xf_1'-\frac{y}{x^2}f_2'+\sin 2x f_3'$

$$\frac{\partial^2 z}{\partial x\partial y}=2x(2yf_{11}''+\frac{1}{x}f_{12}'')-\frac{1}{x^2}f_2'-\frac{y}{x^2}(2yf_{21}''+\frac{1}{x}f_{22}'')+\sin 2x(2yf_{31}''+\frac{1}{x}f_{32}'')$$

(三) 隐函数

① $F(x,y)=0$ — 一个一元函数

$\Downarrow$

$y=\varphi(x)$

② $F(x,y,z)=0$ — 一个二元

若 $F_x'\neq 0$ $\Rightarrow x=\varphi(y,z)$;

若 $F_y'\neq 0$ $\Rightarrow y=\psi(x,z)$;

若 $F_z'\neq 0$ $\Rightarrow z=w(x,y)$;

③ $\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}$ — 二个一元.

$\Downarrow$

$\begin{cases}y=y(x)\\z=z(x)\end{cases}$

型一. 求偏导

P_{135}. 例8. 解:

① 令 $u=\sqrt{x^2+y^2}$ $\frac{\partial z}{\partial x}=\cancel{\frac{x}{\sqrt{x^2+y^2}}f'(u)}\ \frac{x}{u}f'(u)$ $\frac{\partial^2 z}{\partial x^2}=\frac{u-x\frac{x}{u}}{u^2}f'(u)+\frac{x^2}{u^2}f''(u)$

$=\frac{y^2}{u^3}f'(u)+\frac{x^2}{u^2}f''(u)$

$\frac{\partial^2 z}{\partial y^2}=\frac{x^2}{u^3}f'(u)+\frac{y^2}{u^2}f''(u)$

$\Rightarrow \frac{1}{u}f'(u)+f''(u)=0$

(2) $[uf'(u)]' + \frac{1}{u}f'(u) = 0 \qquad f'(u) = C_1 e^{-\int \frac{1}{u}du} = \frac{C_1}{u}$

$\because f'(1) = 1 \quad \therefore C_1 = 1 \quad f'(u) = \frac{1}{u}$

$\Rightarrow f(u) = \ln u + C_2$

$\because f(1) = 0 \quad \therefore C_2 = 0$

$f(u) = \ln u$

P_{135}. 例1. 解：

1° $\begin{cases} y = f(x,t) \\ F(x,y,t) = 0 \end{cases} \Rightarrow \begin{cases} y = y(x) \\ t = t(x) \end{cases}$

2° $\begin{cases} \frac{dy}{dx} = f_1' + f_2' \frac{dt}{dx} \\ F_1' + F_2' \frac{dy}{dx} + F_3' \frac{dt}{dx} = 0 \end{cases}$

P_{136}. 例2. 解.

1° $F(x+\frac{z}{y}, y+\frac{z}{x}) = 0 \Rightarrow z = z(x,y)$

2° $\begin{cases} F_1' \cdot (1 + \frac{1}{y}\frac{\partial z}{\partial x}) + F_2' \cdot \frac{x\frac{\partial z}{\partial x} - z\cdot 1}{x^2} = 0 \\ F_1' \cdot \frac{y\frac{\partial z}{\partial y} - z\cdot 1}{y^2} + F_2'(1 + \frac{1}{x}\frac{\partial z}{\partial y}) = 0 \end{cases}$

P_{136}. 例3. 解：1° $\begin{cases} u = f(x,y,z) \\ xe^x - ye^y = ze^z \end{cases} \Rightarrow \begin{cases} z = z(x,y) \\ u = u(x,y) \end{cases}$

2° $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

3° $\begin{cases} \frac{\partial u}{\partial x} = f_1' + f_3' \frac{\partial z}{\partial x} \\ (x+1)e^x = \frac{\partial z}{\partial x}e^z + ze^z\frac{\partial z}{\partial x} \end{cases} \Rightarrow \frac{\partial z}{\partial x} = \frac{x+1}{z+1}e^{x-z}$

例4. $F(x,y,z) = 0$，F 连续可偏导.

$F_x' F_y' F_z' \neq 0$. 求 $\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x}$

解：$\because F_x' \neq 0$，$\therefore$ 由 $F(x,y,z) = 0 \Rightarrow x = \varphi(y,z)$

$F(x,y,z) = 0 \Rightarrow F_x' \cdot \frac{\partial x}{\partial y} + F_y' = 0 \Rightarrow \frac{\partial x}{\partial y} = -\frac{F_y'}{F_x'}$

$\because F_y' \neq 0$，$\therefore$ 由 $F(x,y,z) = 0 \Rightarrow y = \psi(x,z)$

$F(x,y,z) = 0 \Rightarrow F_y' \cdot \frac{\partial y}{\partial z} + F_z' = 0 \Rightarrow \frac{\partial y}{\partial z} = -\frac{F_z'}{F_y'}$

$\because F_z' \neq 0 \quad \therefore$ 由 $F(x,y,z) = 0 \Rightarrow z = \omega(x,y)$

$F(x,y,z) = 0 \Rightarrow F_x' + F_z'\frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x'}{F_z'}$

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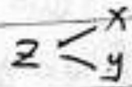
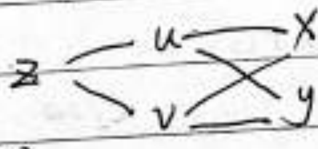
(四) 变换问题:

— **Case 1** $z=f(x,y)$

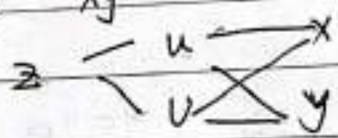


令 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$



P128.例1. 解: $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{x^2+y^2}{xy} \Rightarrow z=z(x,y)$

$\begin{cases} u=x+y \\ v=xy \end{cases}$

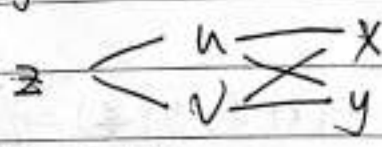


$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}$; $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}$

$$2\frac{\partial z}{\partial u}+u\frac{\partial z}{\partial v}=\frac{u^2-2v}{v}$$

P128 例2. 解: $4\frac{\partial^2 z}{\partial x^2}+12\frac{\partial^2 z}{\partial x\partial y}+5\frac{\partial^2 z}{\partial y^2}=0 \Rightarrow z=z(x,y)$.

$\begin{cases} u=x+ay \\ v=x+by \end{cases}$



$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$, $\frac{\partial z}{\partial y}=a\frac{\partial z}{\partial u}+b\frac{\partial z}{\partial v}$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=a\frac{\partial^2 z}{\partial u^2}+b\frac{\partial^2 z}{\partial u\partial v}+a\frac{\partial^2 z}{\partial v\partial u}+b\frac{\partial^2 z}{\partial v^2}$$

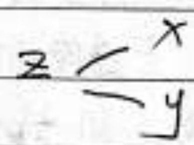
$$=a\frac{\partial^2 z}{\partial u^2}+(a+b)\frac{\partial^2 z}{\partial u\partial v}+b\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2}=a\left(a\frac{\partial^2 z}{\partial u^2}+b\frac{\partial^2 z}{\partial u\partial v}\right)+b\left(a\frac{\partial^2 z}{\partial v\partial u}+b\frac{\partial^2 z}{\partial v^2}\right)$$

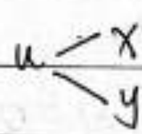
$$=a^2\frac{\partial^2 z}{\partial u^2}+2ab\frac{\partial^2 z}{\partial u\partial v}+b^2\frac{\partial^2 z}{\partial v^2}$$

$$\underbrace{(\qquad)}_{=0}\frac{\partial^2 z}{\partial u^2}+\underbrace{(\qquad\qquad)}_{\neq 0}\frac{\partial^2 z}{\partial u\partial v}+\underbrace{(\qquad)}_{=0}\frac{\partial^2 z}{\partial v^2}=0$$

— **Case 2.** $z=z(x,y)$



$$z=u(x,y)e^{ax+by}$$



$$\frac{\partial z}{\partial x}=\frac{\partial u}{\partial x}e^{ax+by}+au\,e^{ax+by}$$

$$\frac{\partial z}{\partial y}=\frac{\partial u}{\partial y}e^{ax+by}+bu\,e^{ax+by}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 u}{\partial x^2}e^{ax+by}+2a\frac{\partial u}{\partial x}e^{ax+by}+a^2u\,e^{ax+by}$$

(五) 反问题:

P_{137} 例1. 解: $f''_{yy}(x,y)=2x \Rightarrow f'_y(x,y)=2xy+\varphi(x)$

$\because f'_y(x,0)=\sin x$. $\therefore \varphi(x)=\sin x$

$f'_y(x,y)=2xy+\sin x \Rightarrow f(x,y)=xy^2+y\sin x+h(x)$

$\because f(x,1)=0 \quad \therefore x+\sin x+h(x)=0 \Rightarrow h(x)=-x-\sin x$

$f(x,y)=xy^2+y\sin x-x-\sin x$

(六) 代数应用

$\begin{cases}\text{无条件极值} \\ \text{条件极值}\end{cases}$

(一) 无条件极值.

二元: $z=f(x,y)$ 或 $F(x,y,z)=0$.

$(x,y)\in D$ (开)

1° $\begin{cases}\frac{\partial z}{\partial x}=\text{--}=0 \\ \frac{\partial z}{\partial y}=\text{—}=0\end{cases} \Rightarrow \begin{cases}x=? \\ y=?\end{cases}$

2° 设 (x_0,y_0) 为驻点,

$A=f''_{xx}(x_0,y_0) \quad B=f''_{xy}(x_0,y_0) \quad C=f''_{yy}(x_0,y_0)$

$AC-B^2\begin{cases}>0 & \checkmark & \begin{cases}A>0, \text{小} \\ A<0, \text{大}\end{cases} \\ <0 & \times\end{cases}$

P_{138} 例1. 解:

1° $\begin{cases}\frac{\partial z}{\partial x}=4x^3-4y=0 \\ \frac{\partial z}{\partial y}=4y^3-4x=0\end{cases} \Rightarrow \begin{cases}x=0, \\ y=0,\end{cases} \begin{cases}x=-1 \\ y=-1\end{cases}, \begin{cases}x=1 \\ y=1\end{cases}$

2° $\frac{\partial^2 z}{\partial x^2}=12x^2 \quad \frac{\partial^2 z}{\partial x\partial y}=-4 \quad \frac{\partial^2 z}{\partial y^2}=12y^2$

$(x,y)=(0,0)$ 时 $A=0 \quad B=-4 \quad C=0$.

$\because AC-B^2<0 \quad \therefore (0,0)$ 不是极值点.

$(x,y)=(-1,-1)$ 时 $A=12 \quad B=-4 \quad C=12$.

$AC-B^2>0$ 且 $A>0 \quad \therefore (-1,-1)$ 为极小点.

极小值 $z=3$;

$(x,y)=(1,1)$ 时

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P138. 例2. 解: $2x^2+2y^2+z^2+8xz-z+8=0 \Rightarrow z=z(x,y)$.

$$\begin{cases} 4x+2z\frac{\partial z}{\partial x}+8z+8x\frac{\partial z}{\partial x}-\frac{\partial z}{\partial x}=0 \\ 4y+2z\frac{\partial z}{\partial y}+8x\frac{\partial z}{\partial y}-\frac{\partial z}{\partial y}=0 \end{cases}$$

取 $\frac{\partial z}{\partial x}=0$, $\frac{\partial z}{\partial y}=0$.

$$\Rightarrow \begin{cases} x=-2z \\ y=0 \end{cases} \overset{代入}{\Longrightarrow} \begin{cases} x=-2 \\ y=0 \end{cases},\ z=1$$

$\cancel{(8z^2+z^2-16z^2-z+8=0)}$

$$\begin{cases} x=\frac{16}{7} \\ y=0 \end{cases},\ z=-\frac{8}{7}$$

$$\begin{cases} 4+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}+16\frac{\partial z}{\partial x}+8x\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x^2}=0 \\ 2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}+8\frac{\partial z}{\partial y}+8x\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial x\partial y}=0 \\ 4+2\left(\frac{\partial z}{\partial y}\right)^2+2z\frac{\partial^2 z}{\partial y^2}+8x\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial y^2}=0 \end{cases}$$

当 $(x,y)=(-2\ 0)$ 时

$A=\frac{4}{15}$ $B=0$ $C=\frac{4}{15}$

$\because AC-B^2>0$ 且 $A>0$ $\therefore (-2,0)$ 为极小点, 极小值 $z=1$;

(二) 条件极值

如 $z=f(x,y)$. $s.t.\ \varphi(x,y)=0$ (等式)

1° $F=f(x,y)+\lambda\varphi(x,y)$

$$2°\ \begin{cases} F_x'=f_x'+\lambda\varphi_x'=0 & ① \\ F_y'=f_y'+\lambda\varphi_y'=0 & ② \\ F_\lambda'=\varphi(x,y)=0 & ③ \end{cases} \Rightarrow \begin{cases} x=? \\ y=? \end{cases}$$

1. P139. 例3. 解:

① 当 $4x^2+y^2<25$ 时.

$$\begin{cases} z_x'=2x+12y=0 \\ z_y'=12x+4y=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases} \quad z(0,0)=0;$$

② 当 $4x^2+y^2=25$ 时

令 $F=x^2+12xy+2y^2+\lambda(4x^2+y^2-25)$

$$令.\ \begin{cases} F_x'=2x+2y+8\lambda x=0 \\ F_y'=12x+4y+2\lambda y=0 \\ F_\lambda'=4x^2+y^2-25=0 \end{cases}$$

$$\Rightarrow \begin{cases} (1+4\lambda)x+6y=0 \\ 6x+(2+\lambda)y=0 \\ 4x^2+y^2=25 \end{cases}$$

$\therefore \begin{vmatrix} 1+4\lambda & 6 \\ 6 & 2+\lambda \end{vmatrix}=0 \Rightarrow 4\lambda^2+9\lambda-34=0 \Rightarrow (\lambda-2)(4\lambda+17)=0$

$\Rightarrow \lambda_1=2 \quad \lambda_2=-\frac{17}{4}$

当 $\lambda=2$ 时 $y=-\frac{3}{2}x \Rightarrow \begin{cases} x=-2, \\ y=3, \end{cases} \begin{cases} x=2. \\ y=-3. \end{cases}$

当 $\lambda=-\frac{17}{4}$ 时，$y=\frac{8}{3}x \Rightarrow$ ~~$4x^2+\frac{64}{9}x^2=25$~~. $\begin{cases} x=-\frac{3}{2} \\ y=-4 \end{cases} \begin{cases} x=\frac{3}{2} \\ y=4 \end{cases}$

$z(-2,3)= \qquad z(2,-3)=$

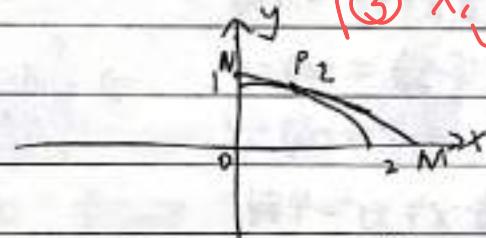
$z(-\frac{3}{2},4)= \qquad z(\frac{3}{2},4)=$

思路 $\begin{cases} ① \\ ② \Rightarrow \lambda=? \text{ 代}\lambda ① \quad y=?x \\ ③ \end{cases}$ 求 λ

① 找到一个方程 x,λ 或 y,λ

② 消去 λ

③ x,y 有非0解 $|\cdot|=0$ 解出 λ



例2. 求三角形面积最小值

解：令 $P(x,y)\in L$

由 $\frac{x^2}{4}+y^2=1 \Rightarrow \frac{x}{2}+2y\cdot y'=0 \Rightarrow y'=-\frac{x}{4y}$

切线. $Y-y=-\frac{x}{4y}(X-x)$

令 $Y=0 \Rightarrow X=x+\frac{4y^2}{x}=\frac{4}{x}(\frac{x^2}{4}+y^2)=\frac{4}{x}$;

令 $X=0 \Rightarrow Y=\frac{x^2}{4y}+y=\frac{1}{y}(\frac{x^2}{4}+y^2)=\frac{1}{y}$;

$S=\frac{2}{xy}$ s.t. $\frac{x^2}{4}+y^2-1=0 \quad (x\geq 0, y\geq 0)$.

$F=\frac{2}{xy}+\lambda(\frac{x^2}{4}+y^2-1)$ 灭 λ.

由 $\begin{cases} F_x'=-\frac{2}{x^2y}+\frac{\lambda}{2}x=0 \\ F_y'=-\frac{2}{xy^2}+2\lambda y=0 \\ F_\lambda'=\frac{x^2}{4}+y^2-1=0 \end{cases} \Rightarrow y=\frac{1}{2}x \Rightarrow \begin{cases} x=\sqrt{2} \\ y=\frac{\sqrt{2}}{2} \end{cases}$

$S_{min}=\frac{2}{\sqrt{2}\cdot\frac{\sqrt{2}}{2}}=2$

Date. /　　　　例一4

如: $\begin{cases} x+\lambda x=0 \ \checkmark \\ 2y-\lambda y=0 \\ x^2+4y^2-4=0 \end{cases}$　　$\begin{cases} x=0 \Rightarrow \begin{cases} x=0 \\ y=\pm 1 \end{cases} \\ \lambda=-1 \Rightarrow y=0 \Rightarrow \begin{cases} x=\pm 2 \\ y=0 \end{cases} \end{cases}$

P139 例4. 解:

① 法一: $du=2x\,dx-2y\,dy=d(x^2-y^2)$

$\Rightarrow u=x^2-y^2+C$

$\because u(0,0)=3 \quad \therefore C=3 \quad u=x^2-y^2+3$

法二: $\frac{\partial u}{\partial x}=2x \quad \frac{\partial u}{\partial y}=-2y$

$\frac{\partial u}{\partial x}=2x \Rightarrow u=x^2+\varphi(y)$

$\frac{\partial u}{\partial y}=\varphi'(y)=-2y \Rightarrow \varphi(y)=-y^2+C$

$\therefore u=x^2-y^2+C$

$\because u(0,0)=3 \quad \therefore C=3 . \quad u=x^2-y^2+3$

② 当 $x^2+4y^2<4$ 时

由 $\begin{cases} \frac{\partial u}{\partial x}=2x=0 \\ \frac{\partial u}{\partial y}=-2y=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}, \quad u(0,0)=\underline{3}.$

当 $x^2+4y^2=4$ 时

令 $\begin{cases} x=2\cos t \\ y=\sin t \end{cases} \quad (0\le t\le 2\pi)$

$u=4\cos^2 t-\sin^2 t+3 \quad =5\cos^2 t+2.$

当 $\cos t=0$ 时. $\quad u_{min}=2$

当 $\cos t=\pm 1$ 时 $\quad u_{max}=7$

$\therefore m=2 \quad M=7$

If $u=f(x,y,z)$. s.t. $\begin{cases} \varphi(x,y,z)=0 \\ \psi(x,y,z)=0 \end{cases}$

$F=f+\lambda\varphi+\mu\psi$　$\begin{cases} F'_x=f'_x+\lambda\varphi'_x+\mu\psi'_x=0 \\ F'_y= \ \dots \\ F'_z= \ \dots \end{cases}$　$F'_\lambda=\varphi=0$ 　$F'_\mu=\psi=0$

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第三模块 积分学

Part I. 不定积分

一、def $\int f(x)dx = F(x)+C$

$f(x)$ 有第一类间断点的函数 无原函数

二　　可能有

二、工具.

(一) 公式

① $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+C$

② $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a}+C$

③ $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2})+C$

④ $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|+C$

⑤ $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$

⑥ $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$

(二) 积分法.

1. 换元法 $\begin{cases}\text{第一类}\\ \text{第二类 ✓}\end{cases}$

第一类：例1. $\int \frac{x}{(2x+3)^3}dx = \frac{1}{4}\int \frac{2x+3-3}{(2x+3)^3}d(2x+3) = \frac{1}{4}\int\frac{d(2x+3)}{(2x+3)^2} - \frac{3}{4}\int(2x+3)^{-3}d(2x+3)$

$= -\frac{1}{4(2x+3)}+\frac{3}{8(2x+3)^2}+C$

例2. $\int \frac{1+x}{x(2+xe^x)}dx = \int\frac{d(xe^x)}{xe^x(xe^x+2)} = \frac{1}{2}\ln\left|\frac{xe^x}{xe^x+2}\right|+C$

第二类：Case1. 平方和差：

1) $\sqrt{a^2-x^2} \xrightarrow{x=a\sin t} a\cos t$

2) $\sqrt{x^2+a^2} \xrightarrow{x=a\tan t} a\sec t$

3) $\sqrt{x^2-a^2} \xrightarrow{x=a\sec t} a\tan t$

2. 分部法：$\int u\,dv = uv-\int v\,du$

例1. $\int\frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int\frac{(2x+1)+1}{x^2+x+1}dx = \frac{1}{2}\ln(x^2+x+1)+\frac{1}{2}\int\frac{d(x+\frac{1}{2})}{(\frac{\sqrt{3}}{2})^2+(x+\frac{1}{2})^2}$

例2. $\int\frac{xe^x}{(x+1)^2}dx = \int\frac{[(x+1)-1]e^x}{(x+1)^2}dx = \int\frac{e^x}{x+1}dx+\int e^x d(\frac{1}{x+1})$

$= \int\frac{e^x}{x+1}dx + \frac{e^x}{x+1} - \int\frac{e^x}{x+1}dx$

$= \frac{e^x}{x+1}+C$

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3. $\int \frac{1+\sin x}{1+\cos x} e^x dx = \int \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} e^x dx = \int \frac{1}{2}\sec^2\frac{x}{2} \cdot e^x dx + \int e^x \tan\frac{x}{2} dx$

$= \int e^x d(\tan\frac{x}{2}) + \int e^x \tan\frac{x}{2} dx$

$= e^x \tan\frac{x}{2} + C$

4. $\int \frac{dx}{\sqrt{x(x+1)}} = 2\int \frac{d(\sqrt{x})}{\sqrt{(\sqrt{x})^2+1}} = 2\ln(\sqrt{x}+\sqrt{x+1}) + C$

$\int R(x)\,dx$

1° If $R(x)$ 假 $\Longrightarrow R(x) =$ 多+真

2° 真分式 $= \frac{\text{分子不变}}{\text{因式分解}} \Rightarrow$ 拆部分和

① $\frac{3x+2}{(2x+1)(x-2)} = \frac{A}{2x+1} + \frac{B}{x-2}$

② $\frac{x^2-x+2}{(x-1)^2(2x-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x-1}$

③ $\frac{x^2-3x+5}{(x-1)(x^2+2x+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+4}$

例1. $\int \ln(1+\sqrt{\frac{x+1}{x}})\,dx$

解 $\sqrt{\frac{x+1}{x}} = t \Rightarrow x = \frac{1}{t^2-1}$

原式 $= \int \ln(1+t)\, d(\frac{1}{t^2-1})$

$= \frac{1}{t^2-1}\ln(1+t) - \int \frac{1}{(t+1)^2(t-1)} dt$

$\frac{1}{(t+1)^2(t-1)} = \frac{A}{t+1} + \frac{B}{(t+1)^2} + \frac{C}{t-1}$

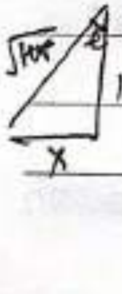
$A(t^2-1) + B(t-1) + C(t^2+2t+1) = 1$

$$\begin{cases} A+C=0 \\ B+2C=0 \\ -A-B+C=1 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{4} \\ B=-\frac{1}{2} \\ C=\frac{1}{4} \end{cases}$$

原式 $= \frac{1}{t^2-1}\ln(1+t) - \frac{1}{4}\ln(t+1) + \frac{1}{2(t+1)} + \frac{1}{4}\ln|t-1| + C$

$= \frac{1}{t^2-1}\ln(1+t) + \frac{1}{4}\ln\frac{t-1}{t+1} + \frac{1}{2(t+1)} + C$

例2. $\int \frac{e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}} dx \xlongequal{x=\tan t} \int \frac{e^t}{\sec^3 t} \times \sec^2 t\, dt = \int e^t \cos t\, dt = I$



$= \int e^t d(\sin t) = e^t \sin t - \int e^t \sin t\, dt = e^t \sin t + \int e^t d(\cos t)$

$= e^t \sin t + e^t \cos t - \int e^t \cos t\, dt$

原式 $= \frac{1}{2}e^t(\sin t + \cos t) + C = \frac{1}{2}e^{\arctan x} \cdot \frac{x+1}{\sqrt{1+x^2}} + C$

例3. $\int \arctan\sqrt{e^x-1}\,dx$

解: $\sqrt{e^x-1}=t \Rightarrow x=\ln(1+t^2)$

原 $=\int \arctan t\, d(\ln(1+t^2)) = \arctan t\cdot \ln(1+t^2) - \int \frac{\ln(1+t^2)}{1+t^2}\,dt$.

分段函数:

$$f(x)=\begin{cases}\frac{1}{1+4x^2}, & x\le 0\\ e^{2x}, & x>0\end{cases} \qquad \int f(x)\,dx\ ?$$

$$\int f(x)\,dx = \begin{cases}\frac{1}{2}\arctan 2x + C_1, & x\le 0\\ \frac{1}{2}e^{2x}+C_2, & x>0\end{cases}$$

取 $C_1=C$ 由 $C=\frac{1}{2}+C_2 \Rightarrow C_2=C-\frac{1}{2}$

$$\int f(x)\,dx = \begin{cases}\frac{1}{2}\arctan 2x + C, & x\le 0\\ \frac{1}{2}e^{2x}+C-\frac{1}{2}, & x>0\end{cases}$$

Part II 定积分及应用

一、def

Notes:

① 可积与∃原函数不同

If $\lim\limits_{\lambda\to 0}\sum f(\xi_i)\Delta x_i$ ∃, $f(x)$在$[a,b]$可积

$f(x)$, $F(x)$, If $F'(x)=f(x)$

② 若$f(x)$连续或∃有限个第一类间断点,则可积.

如: $f(x)=\begin{cases}\arctan x, & x>0\\ \frac{1}{1+x^2}, & x\le 0\end{cases}$

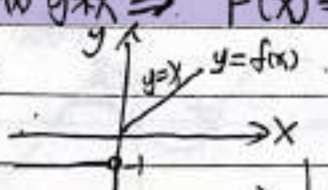
$$\int_0^2 f(x-1)\,dx = \int_0^2 f(x-1)\,d(x-1) = \int_{-1}^1 f(x)\,dx$$

$$= \int_{-1}^0 \frac{1}{1+x^2}\,dx + \int_0^1 \arctan x\,dx$$

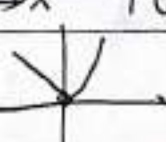
③ $f(x)$可积 $\Rightarrow$ $F(x)=\int_a^x f(t)\,dt$ 连续 ($f(x)$连续, $F(x)$才可导)

不一定可导

如:



$$F(x)=\int_0^x f(t)\,dt = \begin{cases}-x, & x<0\\ \frac{1}{2}x^2, & x\ge 0\end{cases}$$



$F'_-(0)=-1$

$F'_+(0)=0$

④ 若 $f(x)$ 奇函数，$F(x)=\int_a^x f(t)\,dt$ 偶函数

$$F(-x)=\int_a^{-x} f(t)\,dt \xlongequal{t=-u} \int_{-a}^{x} f(-u)(-du) = \int_{-a}^{x} f(u)\,du = \underbrace{\int_{-a}^{a} f(u)\,du}_{0} + \underbrace{\int_a^x f(u)\,du}_{F(x)}$$

如：$\int_a^x t^2[f(t)-f(-t)]\,dt$ 偶函数

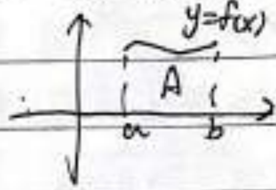
若 $f(x)$ 偶函数，$F(x)=\int_a^x f(t)\,dt$ 不一定奇函数。

如 $f(x)=x^2$ $F(x)=\int_1^x x^2\,dx=\frac{x^3}{3}-\frac{1}{3}$

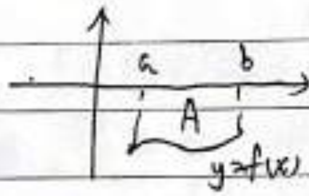
但 $F(x)=\int_0^x f(t)\,dt$ 奇函数。

$$F(-x)=\int_0^{-x} f(t)\,dt \xlongequal{t=-u} \int_0^x f(-u)(-du) = -\int_0^x f(u)\,du=-F(x)$$

⑤



$\int_a^b f(x)\,dx=A$



$\int_a^b f(x)\,dx=-A$

例1. 证明 $x\geq 0$ 时 $f(x)=\int_0^x (t-t^2)\sin^2 t\,dt\leq\frac{1}{20}$

证：$f'(x)=(x-x^2)\sin^2 x \Rightarrow x=1,\ k\pi\ (k=1,2,\dots)$

当 $0<x<1$ 时 $f'(x)>0$

当 $x>1$ 时 $f'(x)\leq 0$.

$\therefore x=1$ 为 $f(x)$ 最大点

$$M=f(1)=\int_0^1 (t-t^2)\sin^2 t\,dt$$

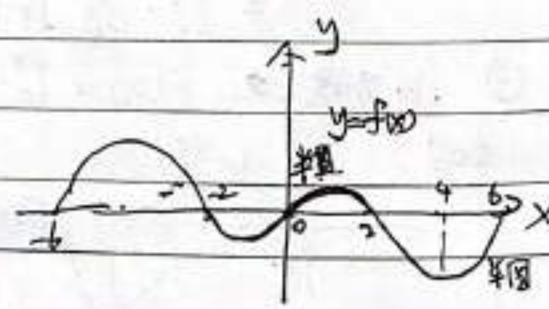
$\because 0\leq\sin t\leq t\quad(0\leq t\leq 1)$

$$\therefore M\leq\int_0^1(t-t^2)t^2\,dt=\int_0^1(t^3-t^4)\,dt=\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{1}{20}$$

$\therefore f(x)\leq\frac{1}{20}$

例. $F(x)=\int_0^x f(t)\,dt$.

$F(-6)$ 与 $F(-4)$ 关系?



解: $F(-6)=F(6)$ $F(-4)=F(4)$.

$F(4)=\frac{\pi}{2}-\pi$ $F(6)=\frac{\pi}{2}-2\pi$

二. 基本定理.

√Th1 $f(x)\in C[a,b]$. $\Phi(x)=\int_a^x f(t)dt$, $\Phi'(x)=f(x)$

Th2. (N-L). $\int_a^b f(x)dx=F(b)-F(a)$

证: $\Phi(x)=\int_a^x f(t)dt$. $\Phi'(x)=f(x)$

$F'(x)=f(x)$

$\Rightarrow F(x)-\Phi(x)\equiv C_0$

$\Rightarrow F(a)-\Phi(a)=F(b)-\Phi(b)$

$\because \Phi(a)=0$, $\therefore \Phi(b)=F(b)-F(a)$

$\therefore \int_a^b f(x)dx=F(b)-F(a)$

例1. $f(x)$: ① $f'(x)+f(x)-\frac{1}{x+1}\int_0^x f(t)dt=0$

② $f(0)=1$;

①求 $f'(x)$; ②证 $x\geq0$ 时, $e^{-x}\leq f(x)\leq1$.

解: ① $(x+1)f'(x)+(x+1)f(x)-\int_0^x f(t)dt=0$

$\Rightarrow f'(x)+(x+1)f''(x)+\cancel{f(x)}+(x+1)f'(x)-\cancel{f(x)}=0$

$\Rightarrow (x+1)f''(x)+(x+2)f'(x)=0$

$\Rightarrow (f'(x))'+(1+\frac{1}{x+1})f'(x)=0$

$f'(x)=Ce^{-\int(1+\frac{1}{x+1})dx}=\frac{Ce^{-x}}{x+1}$

$\because f(0)=1$ $\therefore f'(0)=-1$ $\therefore C=-1$

$\therefore f'(x)=-\frac{e^{-x}}{x+1}$

f.f': $\begin{cases} f'>0(<0)\Rightarrow f\uparrow(\downarrow) \\ f(x)-f(a)=f'(\xi)(x-a) \\ f(x)-f(a)=\int_a^x f'(t)dt \end{cases}$

② $\because f'(x)<0$ $(x\geq0)$ $\therefore f(x)\downarrow$.

$\therefore x\geq0$ 时. $f(x)\leq f(0)=1$

$f(x)-f(0)=\int_0^x f'(t)dt=-\int_0^x \frac{e^{-t}}{t+1}dt\geq-\int_0^x e^{-t}dt=e^{-t}\Big|_0^x=e^{-x}-1$

$\therefore f(x)\geq e^{-x}$

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三 性质:

(一) 一般性:

1、① (积分中值定理) 设 $f(x) \in C[a,b]$，则 $\exists \xi \in [a,b]$，使

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

证: $f(x) \in C[a,b] \Rightarrow \exists m, M.$

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

$$\Rightarrow m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$$

$\exists \xi \in [a,b]$，使 $f(\xi) = \frac{1}{b-a}\int_a^b f(x)dx.$

Note: $f(x) \in C[a,b]$，$f(x)$ 在 $[a,b]$ 平均值为

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

② (推广) 设 $f(x) \in C[a,b]$，则 $\exists \xi \in (a,b)$，使

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

证: $F(x) = \int_a^x f(t)dt \qquad F'(x) = f(x)$

$$\int_a^b f(x)dx = F(b) - F(a) = F'(\xi)(b-a) = f(\xi)(b-a) \quad (a<\xi<b)$$

如: $f(x) \in C[a,b]$，(a,b) 内可导，

$f(a) = 0$，$\int_a^b f(x)dx = 0$. 证 $\exists \xi \in (a,b)$，$f'(\xi) = 0$.

$F(x) = \int_a^x f(t)dt$，$\quad F'(x) = f(x)$

$F(a) = F(b) = 0$

$\exists c \in (a,b) \quad F'(c) = 0 \Rightarrow f(c) = 0$

$f(a) = f(c) = 0$

$\exists \xi \in (a,c) \subset (a,b)$，$f'(\xi) = 0.$

③ 积分第一中值定理 $f(x), g(x) \in C[a,b]$，$g(x) \ge 0$

则 $\exists \xi \in [a,b]$. $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$

证: Case 1 $\quad g(x) \equiv 0$，不证

Case 2. $g(x) \ge 0$ 但 $g(x) \not\equiv 0 \Rightarrow \int_a^b g(x)dx > 0$

$f(x) \in C[a,b] \Rightarrow \exists m, M$

$$m \le f(x) \le M$$

$$\Rightarrow mg(x) \le f(x)g(x) \le Mg(x)$$

$$\Rightarrow \quad m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx$$

$$\Rightarrow \quad m \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le M$$

$\exists \xi \in [a,b]$, 使 $f(\xi) =$

2. ① $\begin{cases} f(x) \in C[a,b] \\ f(x) \ge 0 \\ f(x) \not\equiv 0 \end{cases} \Rightarrow \int_a^b f(x)dx > 0$

② $\begin{cases} f(x), g(x) \in C[a,b] \\ f(x) \ge g(x) \\ f(x) \not\equiv g(x) \end{cases} \Rightarrow \int_a^b f(x)dx > \int_a^b g(x)dx$

例1. $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx \ (n \ge 2)$

① $I_n + I_{n+2}$; ② 证 $\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$

解: ① $I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^n x \, d(\tan x) = \frac{1}{n+1} \tan^{n+1} x \Big|_0^{\frac{\pi}{4}} = \frac{1}{n+1}$

② $\begin{cases} \tan^n x, \ \tan^{n+2} x \in C[0, \frac{\pi}{4}] \\ \tan^n x \ge \tan^{n+2} x \\ \tan^n x \not\equiv \tan^{n+2} x \end{cases} \Rightarrow I_n > I_{n+2}$

$\frac{1}{n+1} = I_n + I_{n+2} < 2I_n \Rightarrow I_n > \frac{1}{2(n+1)}$

$\begin{cases} I_{n-2} + I_n = \frac{1}{n-1}; \\ I_{n-2} > I_n \end{cases}$

$\therefore \frac{1}{n-1} = I_{n-2} + I_n > 2I_n \Rightarrow I_n < \frac{1}{2(n-1)}$

例2. $I_1 = \int_{-1}^1 \frac{(1+x)^2}{1+x^2} dx \quad I_2 = \int_{-1}^1 \frac{1+x}{e^x} dx$ 大小.

解: $I_1 = \int_{-1}^1 1 \, dx$ 由

又 $e^x \ge 1+x$ 且 "=" $\Leftrightarrow$ "$x=0$"

$\Rightarrow \frac{1+x}{e^x} \le 1$

$\begin{cases} 1, \frac{1+x}{e^x} \in C[-1,1]. \\ \frac{1+x}{e^x} \le 1 \\ \frac{1+x}{e^x} \not\equiv 1 \end{cases} \Rightarrow \int_{-1}^1 \frac{1+x}{e^x} dx < \int_{-1}^1 1 \, dx$

$I_2 < I_1$

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3. 设 $f(x), g(x) \in C[a,b]$. 则

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx$$

证: Case 1. $f(x) \equiv 0$ 不证

Case 2. $f(x) \not\equiv 0$

$\forall t \in R \quad [tf(x)+g(x)]^2 \ge 0$

$t^2 f^2(x) + 2tf(x)g(x) + g^2(x) \ge 0$

$$\underbrace{\left(\int_a^b f^2(x)dx\right)}_{A}t^2 + \underbrace{\left(2\int_a^b f(x)g(x)dx\right)}_{B}t + \underbrace{\int_a^b g^2(x)dx}_{C} \ge 0$$

$$\because \begin{cases} f^2(x) \in C[a,b] \\ f^2(x) \ge 0 \\ f^2(x) \not\equiv 0 \end{cases}, \quad \therefore \int_a^b f^2(x)dx > 0$$

$$\therefore \Delta = 4\left(\int_a^b f(x)g(x)dx\right)^2 - 4\int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx \le 0$$

(二) 特殊性质.

1. $\int_{-a}^{a} f(x)dx = \int_0^a [f(x)+f(-x)]dx$

例1. $F(x) = \int_x^{x+2\pi} e^{\sin t} \cdot \sin t\, dt$ ( )

(A) $F(x)$ 与 x 有关 (B) 正常数 (C) 负常数 (D) 为0.

解: $F(x) = \int_{-\pi}^{\pi} e^{\sin t}\sin t\,dt = \int_0^{\pi} (e^{\sin t}\sin t - e^{-\sin t}\sin t)\,dt$

$= \int_0^{\pi} (e^{\sin t} - e^{-\sin t})\sin t\,dt$

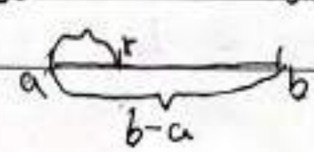
$$\because \begin{cases} (e^{\sin t}-e^{-\sin t})\sin t \in C[0,\pi] \\ (e^{\sin t}-e^{-\sin t})\sin t \ge 0 \\ (e^{\sin t}-e^{-\sin t})\sin t \not\equiv 0 \end{cases}, \quad \therefore F(x) > 0$$

2. ① $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$

$\int_a^b \xlongequal{x+t=a+b} \int_a^b$

$\int_a^b f(x)dx \xlongequal{x+t=a+b} \int_b^a f(a+b-t)(-dt) = \int_a^b f(a+b-x)dx$

$\int_a^b \xlongequal{x=a+(b-a)t} \int_0^1$



证：左 $\xlongequal{x+t=\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{0} f(\cos t)(-dt) = \int_0^{\frac{\pi}{2}} f(\cos t)\,dt = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$

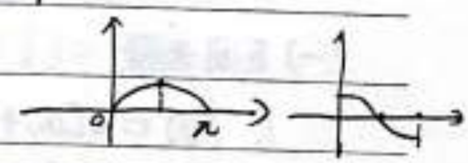
记 $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx \triangleq I_n$

$$\begin{cases} I_n = \frac{n-1}{n} I_{n-2} \\ I_0 = \frac{\pi}{2} \\ I_1 = 1 \end{cases}$$

例2. $I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx$

解：$I = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx \Rightarrow 2I = \frac{\pi}{2} \quad I = \frac{\pi}{4}$

② $$\begin{cases} \int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} f(\sin x)\,dx \\ \int_0^{\pi} f(|\cos x|)\,dx = 2\int_0^{\frac{\pi}{2}} f(\cos x)\,dx \end{cases}$$



证：$\int_0^{\pi} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,dx$

$\int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,dx \xlongequal{x-\frac{\pi}{2}=t} \int_0^{\frac{\pi}{2}} f(\cos t)\,dt = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

③ $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$ $\qquad \cos^{2n} x$ 和 $|\cos x|$

证：$I = \int_0^{\pi} x f(\sin x)\,dx \xlongequal{x+t=\pi} \int_{\pi}^{0} (\pi - t) f(\sin t)(-dt)$

$= \int_0^{\pi} (\pi - t) f(\sin t)\,dt = \int_0^{\pi} (\pi - x) f(\sin x)\,dx$

$= \pi \int_0^{\pi} f(\sin x)\,dx - I$

$I = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\,dx$

例3. $\int_0^{\pi^2} \sin^4 \sqrt{x}\,dx$

解：$\int_0^{\pi^2} \sin^4\sqrt{x}\,dx \xlongequal{\sqrt{x}=t} \int_0^{\pi} \sin^4 t \cdot 2t\,dt = 2\int_0^{\pi} t\sin^4 t\,dt$

$= 2 \times \frac{\pi}{2} \int_0^{\pi} \sin^4 t\,dt = 2\pi I_4 = 2\pi \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} =$

例4 $f(x) = \frac{x}{1+\cos^2 x} + \int_{-\pi}^{\pi} f(x)\sin x\,dx$

求 $\int_0^{\pi} f(x)\,dx$

解：令 $\int_{-\pi}^{\pi} f(x)\sin x\,dx = A$

$f(x) = \frac{x}{1+\cos^2 x} + A$

$f(x)\sin x = \frac{x\sin x}{1+\cos^2 x} + A\sin x$

$A = 2\int_0^{\pi} x \frac{\sin x}{1+\cos^2 x}\,dx = \pi \int_0^{\pi} \frac{d(\cos x)}{1+\cos^2 x} = -\pi \arctan \cos x \Big|_0^{\pi}$

$= -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi^2}{2}$

$\therefore f(x) = \frac{x}{1+\cos^2 x} + \frac{\pi^2}{2}$ $\qquad \int_0^{\pi} f(x)\,dx = \int_0^{\pi} x \frac{1}{1+\cos^2 x}\,dx + \frac{\pi^3}{2}$

$= \frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x}\,dx + \frac{\pi^3}{2} = \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos^2 x}\,dx + \frac{\pi^3}{2}$

$= \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x + 1}\,dx + \frac{\pi^3}{2}$

$= \pi \int_0^{\frac{\pi}{2}} \frac{d(\tan x)}{(\sqrt{2})^2 + \tan^2 x} + \frac{\pi^3}{2}$

$= \pi \times \frac{1}{\sqrt{2}} \arctan \frac{\tan x}{\sqrt{2}} \Big|_0^{\frac{\pi}{2}} + \frac{\pi^3}{2}$

$= \frac{\pi}{\sqrt{2}}\left(\frac{\pi}{2} - 0\right) + \frac{\pi^3}{2}$

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3. $f(x)$ 以 T 为周期，

① $\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$

② $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$

四. 广义积分（反常积分）

正常积分 $\begin{cases} ① \text{ 积分区间有限;} \\ ② \text{ } f(x)\text{在积分区间上连续或有限个第一类间断点} \end{cases}$

（一）区间无限

1. $f(x) \in C[a,+\infty)$

def - $\int_a^b f(x)dx = F(b)-F(a)$

$\lim\limits_{b\to+\infty} [F(b)-F(a)] \begin{cases} =A \text{ ， } \int_a^{+\infty} f(x)dx = A \\ \text{无，发散} \end{cases}$

判别法：$\lim\limits_{x\to+\infty} x^\alpha f(x) = C_0 (\neq 0)$ 　$\left(f(x) \sim \frac{C_0}{x^\alpha}, \ f(x) = O\left(\frac{1}{x^\alpha}\right)\right)$

$\begin{cases} \text{收敛}, & \alpha>1 \\ \text{发散}, & \alpha\leq 1 \end{cases}$

如：$\int_0^{+\infty} \frac{dx}{x^2+2x+2} =$

解：$\int_0^{+\infty} \frac{dx}{x^2+2x+2} = \int_0^{+\infty} \frac{d(x+1)}{1+(x+1)^2} = \arctan(x+1)\Big|_0^{+\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

2. $f(x) \in C(-\infty, a]$

def - $\int_b^a f(x)dx = F(a)-F(b)$

$\lim\limits_{b\to-\infty} [F(a)-F(b)] \begin{cases} =A \text{ ， } \int_{-\infty}^a f(x)dx = A \\ \text{无，发散} \end{cases}$

判别法

$\lim\limits_{x\to-\infty} x^\alpha f(x) = C_0 \neq 0 \begin{cases} \text{收敛}, & \alpha>1 \\ \text{发散}, & \alpha\leq 1 \end{cases}$

3. $f(x) \in C(-\infty, +\infty)$

（广义不能随意使用奇偶，要先判断收敛发散）

$\int_{-\infty}^{+\infty} f(x)dx$ 收敛 $\Leftrightarrow \int_{-\infty}^a f(x)dx$ 与 $\int_a^{+\infty} f(x)dx$ 皆收敛

Q $\int_{-\infty}^{+\infty} \frac{x}{1+x^2}dx \overset{?}{\Rightarrow} 0$

对 $\int_0^{+\infty} \frac{x}{1+x^2}dx$. 　$\because \lim\limits_{x\to+\infty} x^1 \frac{x}{1+x^2} = 1$ 且 $\alpha = 1 \leq 1$ $\therefore \int_0^{+\infty} \frac{x}{1+x^2}dx$ 发散 　$\therefore \int_{-\infty}^{+\infty}$ 也发散

$\left(\because \int_0^{+\infty} \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2)\Big|_0^{+\infty} = +\infty \quad \therefore \int_0^{+\infty} \frac{x}{1+x^2}dx \text{ 发散}\right)$

注：Γ-函数

$$\Gamma(\alpha) \triangleq \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$$

$$\begin{cases} \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \\ \Gamma(n+1) = n! \\ \Gamma(\frac{1}{2}) = \sqrt{\pi} \end{cases}$$

如：$\int_0^{+\infty} x^3 e^{-2x} dx = \frac{1}{16}\int_0^{+\infty} (2x)^3 e^{-2x} d(2x) = \frac{1}{16}\int_0^{+\infty} x^3 e^{-x} dx = \frac{1}{16}\Gamma(4) = \frac{3}{8}$

又如：$\int_0^{+\infty} x^4 e^{-x^2} dx = \frac{1}{2}\int_0^{+\infty} (x^2)^{\frac{3}{2}} e^{-x^2} d(x^2) = \frac{1}{2}\int_0^{+\infty} x^{\frac{3}{2}} e^{-x} dx = \frac{1}{2}\Gamma(\frac{3}{2}+1)$

$= \frac{1}{2}\times\frac{3}{2}\times\Gamma(\frac{1}{2}+1) = \frac{3}{4}\times\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{8}\sqrt{\pi}$.

(二) 区间有限

1. $f(x) \in C(a,b]$ 且 $f(a+0) = \infty$

def — $\forall \varepsilon > 0$. $\int_{a+\varepsilon}^{b} f(x)dx = F(b) - F(a+\varepsilon)$

$\lim\limits_{\varepsilon\to 0^+}[F(b) - F(a+\varepsilon)]$ $\begin{cases} = A & \int_a^b f(x)dx = A \\ \text{无} & \text{发散} \end{cases}$

判别法：

$\lim\limits_{x\to a^+} (x-a)^{\alpha} f(x) = C_0$ $\begin{cases} \text{收敛}, & \alpha<1 \\ \text{发散}, & \alpha\geq 1 \end{cases}$

如：$\int_1^2 \frac{1}{x\sqrt{x-1}} dx$

$= 2\int_1^2 \frac{d(\sqrt{x-1})}{1+(\sqrt{x-1})^2} = 2\arctan\sqrt{x-1}\Big|_1^2 = 2(\frac{\pi}{4} - 0) = \frac{\pi}{2}$

又如：$\int_0^1 \ln x\, dx = x\ln x\Big|_0^1 - \int_0^1 x\cdot\frac{1}{x} dx$

$\because \lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \ln(x^x) = 0$

$\therefore \int_0^1 \ln x\, dx = -1$

2. $f(x) \in C[a,b)$. $f(b-0) = \infty$

def — $\forall \varepsilon > 0$ $\int_a^{b-\varepsilon} f(x)dx = F(b-\varepsilon) - F(a)$

$\lim\limits_{\varepsilon\to 0^+}[F(b-\varepsilon) - F(a)]$ $\begin{cases} = A & \int_a^b f(x)dx = A \\ \text{无} & \text{发散} \end{cases}$

判别法

$\lim\limits_{x\to b^-} (b-x)^{\alpha} f(x) = C_0$ $\begin{cases} \text{收敛}, & \alpha<1 \\ \text{发散}, & \alpha\geq 1 \end{cases}$

C_0 不是0　　$(x-1)^2\cdot\frac{1}{\sqrt{x-1}} \to 0$　　$\alpha = 2 > 1$

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如: $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \arcsin x\Big|_0^1 = \frac{\pi}{2}$

3. $f(x) \in C[a,c)\cup(c,b]$, $f(x)\to\infty\ (x\to c)$

$\int_a^b$ 收 $\Leftrightarrow$ $\int_a^c$, $\int_c^b$ 收敛　拆

例1. $\int_0^{+\infty} \frac{dx}{x\sqrt{x+2}}$

若大题 解: $\lim\limits_{x\to+\infty} x^{\frac{3}{2}}\frac{1}{x\sqrt{x+2}} = 1$ 且 $\alpha = \frac{3}{2} > 1$.

要上步骤 $\lim\limits_{x\to 0^+}(x-0)^1\frac{1}{x\sqrt{x+2}} = \frac{1}{\sqrt{2}}$ 且 $\alpha = 1 \geq 1$

∴ 发散

例2. $\int_0^2 \frac{1}{\sqrt{2x-x^2}}dx$

若大题 解∵ $\lim\limits_{x\to 0^+}(x-0)^{\frac{1}{2}}\frac{1}{\sqrt{2x-x^2}} = \lim\limits_{x\to 0}\frac{1}{\sqrt{2-x}} = \frac{1}{\sqrt{2}}$ 且 $\alpha = \frac{1}{2} < 1$.

又∵ $\lim\limits_{x\to 2^-}(2-x)^{\frac{1}{2}}\frac{1}{\sqrt{2x-x^2}} = \lim\limits_{x\to 2^-}\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{2}}$ 且 $\alpha = \frac{1}{2} < 1$

∴ 收敛

∴ $\int_0^2 \frac{dx}{\sqrt{2x-x^2}} = \int_0^2 \frac{d(x-1)}{\sqrt{1-(x-1)^2}} = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi$

例3. $\int_0^{+\infty} \frac{1}{x^{2-\alpha}(1+x^{3-\alpha})}dx$ 收敛

求 α 范围.

解: $\lim\limits_{x\to 0^+}(x-0)^{2-\alpha}\frac{1}{x^{2-\alpha}(1+x^{3-\alpha})} = 1 \Rightarrow 2-\alpha < 1 \Rightarrow \alpha > 1$

$\lim\limits_{x\to+\infty} x^{5-2\alpha}\frac{1}{x^{2-\alpha}(1+x^{3-\alpha})} = 1 \Rightarrow 5-2\alpha > 1 \Rightarrow \alpha < 2 \Rightarrow 1<\alpha<2$

例4. $\int_2^{+\infty} \frac{dx}{(x-1)^5\sqrt{x^2-2x}}$?

解 ∵ $\lim\limits_{x\to 2^+}(x-2)^{\frac{1}{2}}\frac{1}{(x-1)^5\sqrt{x^2-2x}} = \lim\limits_{x\to 2^+}\frac{1}{(x-1)^5\sqrt{x}} = \frac{1}{\sqrt{2}}$ 且 $\alpha = \frac{1}{2} < 1$

又∵ $\lim\limits_{x\to+\infty} x^6\frac{1}{(x-1)^5\sqrt{x^2-2x}} = 1$ 且 $\alpha = 6 > 1$

∴ 收敛

$\int_2^{+\infty} \frac{dx}{(x-1)^5\sqrt{x^2-2x}} = \int_2^{+\infty} \frac{d(x-1)}{(x-1)^5\sqrt{(x-1)^2-1}} = \int_1^{+\infty} \frac{dx}{x^5\sqrt{x^2-1}}$

$\xlongequal{x=\sec t} \int_0^{\frac{\pi}{2}} \frac{\sec t\tan t}{\sec^5 t\tan t}dt = \int_0^{\frac{\pi}{2}} \cos^4 t\,dt = \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{3\pi}{16}$

例5. $\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{\sqrt{|x-x^2|}}$

解: 1° $I = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{\sqrt{|x-x^2|}} = \int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x-x^2}} + \int_1^{\frac{3}{2}} \frac{dx}{\sqrt{x^2-x}} = I_1 + I_2$

2° ∵ $\lim\limits_{x\to 1^-}(1-x)^{\frac{1}{2}}\frac{1}{\sqrt{x-x^2}} = 1$ 且 $\alpha = \frac{1}{2} < 1$

∴ I_1 收敛

又∵ $\lim\limits_{x\to 1^+}(x-1)^{\frac{1}{2}}\frac{1}{\sqrt{x^2-x}} = 1$ 且 $\alpha = \frac{1}{2} < 1$

∴ I_2 收敛

$3°\ I_1 = 2\int_{\frac{1}{2}}^{1} \frac{d(\sqrt{x})}{\sqrt{1-(\sqrt{x})^2}} = 2\arcsin\sqrt{x}\Big|_{\frac{1}{2}}^{1} = 2(\frac{\pi}{2}-\frac{\pi}{4}) = \frac{\pi}{2}$

$I_2 = 2\int_{1}^{\frac{3}{2}} \frac{d(\sqrt{x})}{\sqrt{(\sqrt{x})^2-1}} = 2\ln\left|x+\sqrt{x-1}\right|\Big|_{1}^{\frac{3}{2}}$

五. 几何应用.

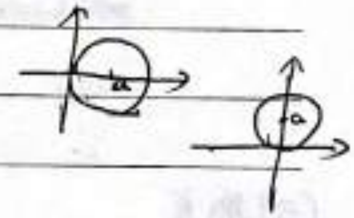
(一) 面积

Notes:

1、圆： ① $x^2+y^2=a^2 \Leftrightarrow r=a$

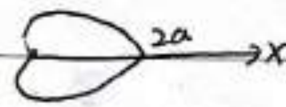
② $x^2+y^2=2ax \Leftrightarrow r=2a\cos\theta$

③ $x^2+y^2=2ay \Leftrightarrow r=2a\sin\theta$



2. 心形线：

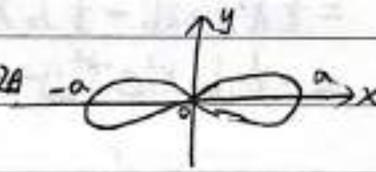
① $r=a(1+\cos\theta)$



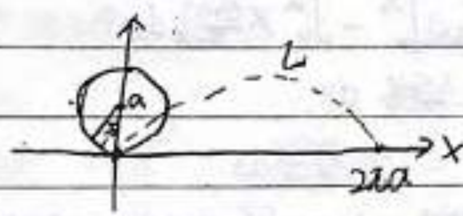
② $r=a(1-\cos\theta)$

3、双纽线

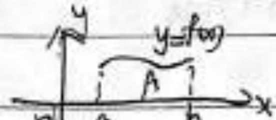
$(x^2+y^2)^2=a^2(x^2-y^2) \Leftrightarrow r^2=a^2\cos 2\theta$

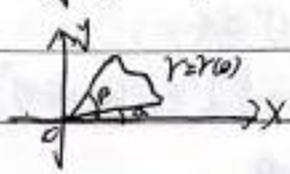


4. 摆线

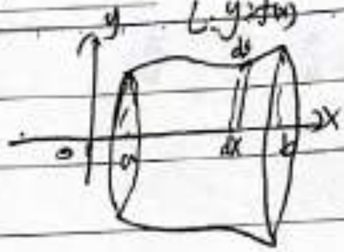


$L:\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases} \quad (0\le t\le 2\pi)$

1、  $A=\int_a^b f(x)dx$

2、  $A=\frac{1}{2}\int_\alpha^\beta r^2(\theta)d\theta$

3.



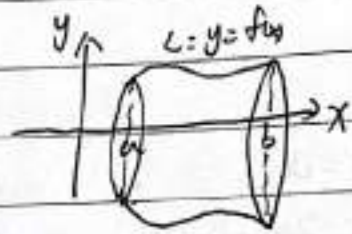
1° $[x, x+dx] \subset [a,b]$

2° $dA = 2\pi|y|\cdot ds = 2\pi\cdot|f(x)|\cdot\sqrt{1+f'^2(x)}\,dx$

3° $A = 2\pi\int_a^b |f(x)|\sqrt{1+f'^2(x)}\,dx$

(二) 体积

1.



$V_x = \pi\int_a^b y^2\,dx$

⋮

(三) 弧长

型一 计算

~~P105~~ P105

例1 解: $\int_0^1 x^2 f(x)\,dx = \int_0^1 f(x)\,d(\frac{1}{3}x^3)$ $\quad f(x)=\int_1^x e^{-x^2}dx$

$=\frac{1}{3}x^3 f(x)\Big|_0^1 - \frac{1}{3}\int_0^1 x^3 e^{-x^2}dx$

$= -\frac{1}{6}\int_0^1 x^2 e^{-x^2}dx^2 = -\frac{1}{6}\int_0^1 x e^{-x}dx$

例2. $\int_0^\pi f(x)\,dx = xf(x)\Big|_0^\pi - \int_0^\pi x\frac{\sin x}{\pi-x}dx$

$=\pi f(\pi) - \int_0^\pi \frac{x\sin x}{\pi-x}dx$

$=\pi\int_0^\pi \frac{\sin x}{\pi-x}dx - \int_0^\pi\frac{x\sin x}{\pi-x}dx$

$=\int_0^\pi \frac{\pi\sin x - x\sin x}{\pi-x}dx = \int_0^\pi \sin x\,dx = 2x| = 2$

例3. $f(x)=\int_0^x \arctan(t-1)^2\,dt$

求 $\int_0^1 f(x)\,dx$

解: $\int_0^1 f(x)\,dx = xf(x)\Big|_0^1 - \int_0^1 x\arctan(x-1)^2dx$

$= f(1) - \int_0^1 x\arctan(x-1)^2\,dx$

$=\int_0^1\arctan(x-1)^2dx - \int_0^1 x\arctan(x-1)^2dx$

$= -\int_0^1 (x-1)\arctan(x-1)^2\,d(x-1) = -\frac{1}{2}\int_0^1 \arctan(x-1)^2\,d[(x-1)^2]$

$= -\frac{1}{2}\int_1^0 \arctan x\,dx = \frac{1}{2}\int_0^1\arctan x\,dx$

抽象函数积分：f(x)表达式未知

例 $\int_0^2 f(x)dx=4$ $\quad f(2)=1$ $\quad f'(2)=0$

$\int_0^1 x^2 f''(2x)dx = \frac{1}{8}\int_0^2 x^2 f''(x)dx = \frac{1}{8}\int_0^2 x^2 d f'(x)$

$=\frac{1}{8}[x^2 f'(x)|_0^2 - 2\int_0^2 x f'(x)dx] = -\frac{1}{4}\int_0^2 x\, df(x)$

常规：

1. $\int_0^{\pi} x\sqrt{\sin^2x-\sin^4x}\,dx = \frac{\pi}{2}\times 2\int_0^{\frac{\pi}{2}}\sqrt{\sin^2x-\sin^4x}\,dx = \pi\int_0^{\frac{\pi}{2}}\sin x\, d(\sin x)$

$=\frac{\pi}{2}\sin^2x|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

2. $\int_{-2}^{2}\left(x^2\sqrt{2x-x^2}+\ln(x+\sqrt{1+x^2})\right)dx$

$=2\int_0^2 x^2\sqrt{2x-x^2}\,dx$ 偶??

$=2\int_0^2 [1+(x-1)]^2\sqrt{1-(x-1)^2}\,d(x-1)$

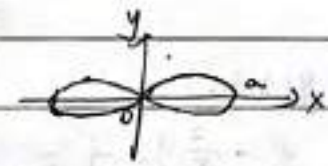
$=2\int_{-1}^{1}(1+x)^2\sqrt{1-x^2}\,dx$

$=4\int_0^1(1+x^2)\sqrt{1-x^2}\,dx \xlongequal{x=\sin t} 4\int_0^{\frac{\pi}{2}}(1+\sin^2 t)(1-\sin^2 t)\,dt$

$=4(\frac{\pi}{2}-\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2})$

题型二. 几何应用

1. 求 $(x^2+y^2)^2=a^2(x^2-y^2)$ 围成面积.

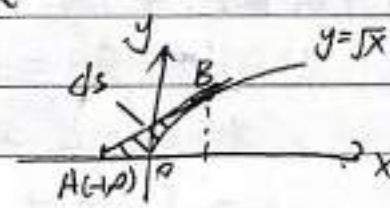


解：L: $r^2=a^2\cos 2\theta$

$A_1=\frac{1}{2}\int_0^{\frac{\pi}{4}} r^2 d\theta$

$=\frac{a^2}{4}\int_0^{\frac{\pi}{4}}\cos 2\theta\, d2\theta = \frac{a^2}{4}$ $\qquad A=a^2$

2. ① A:



② V_x

③ $S_表$

解：① 令 $B(a,\sqrt{a})$

由 $\frac{1}{2\sqrt{a}}=\frac{\sqrt{a}}{a+1} \Rightarrow 2a=a+1 \Rightarrow a=1$

切线 $y-0=\frac{1}{2}(x+1) \Rightarrow y=\frac{1}{2}(x+1)$

$A=\frac{1}{2}\times 2\times 1-\int_0^1\sqrt{x}\,dx = 1-\frac{2}{3}x^{\frac{3}{2}}|_0^1=\frac{1}{3}$

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② $V_x=\pi\int_{-1}^{1}y_2^2dx-\pi\int_0^1 y_1^2dx=\pi\int_{-1}^{1}\frac{1}{4}(x+1)^2dx-\pi\int_0^1 xdx$

$=\frac{\pi}{12}(x+1)^3\Big|_{-1}^{1}-\frac{\pi}{2}=\frac{2\pi}{3}-\frac{\pi}{2}$

③ 取 $[x, x+dx]\subset[-1,1]$

$dS_{外}=2\pi y\,ds=2\pi\times\frac{1}{2}(x+1)\sqrt{1+\frac{1}{4}}\,dx$

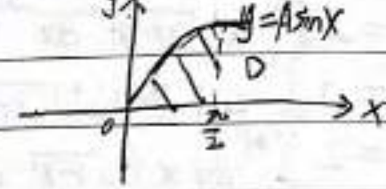
$=\frac{\sqrt{5}}{2}\pi(x+1)dx$

$S_{外}=\frac{\sqrt{5}}{2}\pi\int_{-1}^{1}(x+1)\,dx=\sqrt{5}\pi$

取 $[x, x+dx]\subset[0,1]$

$dS_{内}=2\pi y\cdot ds=2\pi\sqrt{x}\cdot\sqrt{1+\frac{1}{4x}}\,dx=\pi\sqrt{4x+1}\,dx$

$S_{内}=\frac{\pi}{4}\int_0^1(4x+1)^{\frac{1}{2}}d(4x+1)=\frac{\pi}{4}\times\frac{2}{3}(4x+1)^{\frac{3}{2}}\Big|_0^1$



3.

$V_x=V_y$，$A=?$

解：$V_x=\pi\int_0^{\frac{\pi}{2}}y^2dx=\pi A^2\times\frac{1}{2}\times\frac{\pi}{2}=\frac{\pi^2}{4}A^2$

取 $[x, x+dx]\subset[0,\frac{\pi}{2}]$

$dV_y=2\pi x\cdot y\cdot dx=2\pi A\,x\sin x\,dx$

$V_y=2\pi A\int_0^{\frac{\pi}{2}}x\sin x\,dx$

4. L：$y=y(x)$.

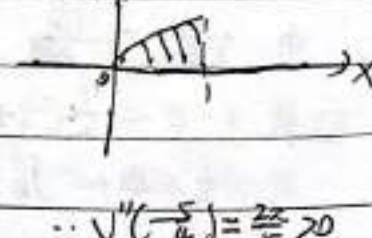
① $y'-\frac{2}{x}y=-1$；$y=y(x)$，$x=0$，$x=1$，x轴围成 D 绕 x轴一周体积最小，求 $y(x)$.

解．1° $y'-\frac{2}{x}y=-1$

$y=[\int-1e^{\int-\frac{2}{x}dx}dx+c\,]e^{-\int-\frac{2}{x}dx}$

$=(\ \frac{1}{x}+c\ )x^2$

$=cx^2+x$



2° $V(c)=\pi\int_0^1 y^2\,dx$

$=\pi\int_0^1(c^2x^4+2cx^3+x^2)dx$

$=\pi(\frac{c^2}{5}+\frac{c}{2}+\frac{1}{3})$

由 $V'(c)=\pi(\frac{2}{5}c+\frac{1}{2})=0$　$c=-\frac{5}{4}$　$\because V''(-\frac{5}{4})=\frac{2\pi}{5}>0$

$\therefore c=-\frac{5}{4}$时 $V(c)$最小.　$\therefore y(x)=x-\frac{5}{4}x^2$

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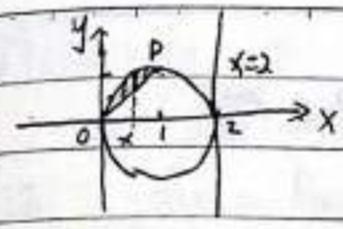
5. 

解: $[x, x+dx] \subset [0,1]$

$dV = 2\pi(2-x)(y_2-y_1)dx = 2\pi(2-x)(\sqrt{2x-x^2}-x)dx$

$V = 2\pi\int_0^1 (2-x)(\sqrt{2x-x^2}-x)dx = 2\pi\int_0^1 (2-x)\sqrt{2x-x^2} - 2\pi(1-\frac{1}{3})$

$= 2\pi\int_0^1 [1-(x-1)]\sqrt{1-(x-1)^2}\,d(x-1) - \frac{4}{3}\pi$

$= 2\pi\int_{-1}^0 (1-x)\sqrt{1-x^2}\,dx - \frac{4}{3}\pi$

$\overset{x=-t}{=} 2\pi\int_1^0 (1+t)\sqrt{1-t^2}\,(-dt) - \frac{4}{3}\pi$

$= 2\pi\int_0^1 (1+x)\sqrt{1-x^2}\,dx - \frac{4}{3}\pi$

$\overset{x=\sin t}{=} 2\pi\int_0^{\frac{\pi}{2}}(1+\sin t)(1-\sin^2 t)dt - \frac{4}{3}\pi$

$= 2\pi\int_0^{\frac{\pi}{2}}(1+\sin t-\sin^2 t-\sin^3 t)dt - \frac{4}{3}\pi$

$= 2\pi(\frac{\pi}{2}+1-\frac{1}{2}\times\frac{\pi}{2}-\frac{2}{3}\times 1) - \frac{4}{3}\pi$

型三 证明

一、f(x)连续

1. $f(x)\in C[a,b]$, $\forall x,y\in[a,b]$ 有

$|f(x)-f(y)|\le 2|x-y|$

$|\int_a^b f(x)dx| \le \int_a^b |f(x)|dx$

证: $|\int_a^b f(x)dx - f(a)(b-a)| \le (b-a)^2$

证: $f(a)(b-a) = \int_a^b f(a)\,dx$

左 $= |\int_a^b [f(x)-f(a)]dx| \le \int_a^b |f(x)-f(a)|dx$

$\le \int_a^b 2|x-a|dx = \int_a^b 2(x-a)d(x-a) = (x-a)^2\Big|_a^b = (b-a)^2$

2. $S(x)=\int_0^x|\cos t|dt$

① 证: 当 $n\pi\le x\le(n+1)\pi$ 时, $2n\le S(x)\le 2(n+1)$;

② $\lim_{x\to+\infty}\frac{S(x)}{x}$

证: ① ∵ $|\cos t|\ge 0$ 　又∵ $n\pi\le x\le(n+1)\pi$

∴ $\int_0^{n\pi}|\cos t|dt \le \int_0^x|\cos t|dt \le \int_0^{(n+1)\pi}|\cos t|dt$

$\int_0^{n\pi}|\cos t|dt = n\int_0^{\pi}|\cos t|dt = 2n\int_0^{\frac{\pi}{2}}\cos t\,dt = 2n$

$\int_0^{(n+1)\pi}|\cos t|dt = 2(n+1)$ 　∴ $2n\le S(x)\le 2(n+1)$

思路 ① 积分限 相同，比较被积函数　③几何面积
② ~~~~ 不同：$\begin{cases} 1) \text{变一样（积分限）} \\ 2) \text{各算各} \end{cases}$

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③ $2n \le S(x) \le 2(n+1)$

$\frac{1}{(n+1)\pi} \le \frac{1}{x} \le \frac{1}{n\pi} \quad \Rightarrow \quad \frac{2n}{(n+1)\pi} \le \frac{S(x)}{x} \le \frac{2(n+1)}{n\pi}$

$\because x \to +\infty \Leftrightarrow n \to +\infty$

且 $\lim\limits_{n\to\infty}$ 左 $= \frac{2}{\pi}$，$\lim\limits_{n\to\infty}$ 右 $= \frac{2}{\pi}$

$\therefore$ 原式 $= \frac{2}{\pi}$

3. $f(x) \in C[0,1]$，$\downarrow$，且 $0 < \alpha < 1$．证：

$$\int_0^\alpha f(x)dx \ge \alpha \int_0^1 f(x)dx$$

证：$\int_0^\alpha f(x)dx \xlongequal{x=\alpha t} \int_0^1 f(\alpha t)\cdot \alpha dt = \alpha \int_0^1 f(\alpha x)dx$

$\because f(x)\downarrow$ 且 $\alpha x \le x$

$\therefore f(\alpha x) \ge f(x)$

$\therefore \int_0^\alpha f(x)dx \ge \alpha \int_0^1 f(x)dx$

~~二、f(x)可导：~~

4. $f(x) \in C[a,b]$，$\uparrow$　证：$\int_a^b x f(x)dx \ge \frac{a+b}{2}\int_a^b f(x)dx$

证，$\varphi(x) = \int_a^x t f(t)dt - \frac{a+x}{2}\int_a^x f(t)dt$　　$\varphi(a) = 0$

$\varphi'(x) = x f(x) - \frac{1}{2}\int_a^x f(t)dt - \frac{a+x}{2}f(x)$

$= \frac{x-a}{2}f(x) - \frac{1}{2}\int_a^x f(t)dt$

$= \frac{x-a}{2}f(x) - \frac{1}{2}f(\xi)(x-a)$

$= \frac{x-a}{2}[f(x) - f(\xi)] \ge 0^{(x>a)}$　$(a \le \xi \le x)$

$\begin{cases} \varphi(a) = 0 \\ \varphi'(x) \ge 0 \ (x>a) \end{cases} \Rightarrow \varphi(x) \ge 0 \ (x>a)$

$\because b > a \quad \therefore \varphi(b) \ge 0$

二、f(x)可导

1° $\begin{cases} f(x) - f(a) = f'(\xi)(x-a) & \text{（积分中无 } f'\text{）} \\ f(x) - f(a) = \int_a^x f'(t)dt & \text{（积分中有 } f'\text{）} \end{cases}$

2° ① |·|：$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$

② $(\ )^2$：$\left(\int_a^b fg\,dx\right)^2 \le \int_a^b f^2dx \int_a^b g^2dx$

③ 无|·|，$(\ )^2$：等式积分．

1. $f'(x) \in C[0,a]$， $f(0)=0$， $|f'(x)| \le M$.

证：$\left|\int_0^a f(x)dx\right| \le \frac{M}{2}a^2$.

证：$f(x) = f(x) - f(0) = f'(\xi)x \quad (0<\xi<x)$

$\left|\int_0^a f(x)dx\right| \le \int_0^a |f(x)|dx \le \int_0^M Mx\,dx = \frac{M}{2}a^2$

2. $f'(x) \in C[0,1]$. $f(0)=0$. 证：$\exists\, \xi \in [0,1]$

使. $f'(\xi) = 2\int_0^1 f(x)dx$

证：$f(x) = f(x) - f(0) = f'(\eta)x \quad (0<\eta<x)$

$\int_0^1 f(x)dx = \int_0^1 f'(\eta)x\,dx$

$\because f'(x) \in C[0,1] \quad \therefore f'(x)$ 在 $[0,1]$ 上有 m, M.

$\therefore \quad \frac{m}{2} \le \int_0^1 f'(\eta)x\,dx \le \frac{M}{2}$

$m \le 2\int_0^1 f(x)dx \le M$.

$\exists\, \xi \in [0,1]$，使 $f'(\xi) = 2\int_0^1 f(x)dx$

3. $f'(x) \in C[a,b]$. $f(a) = f(b) = 0$， $a<c<b$.

证：$|f(c)| \le \frac{1}{2}\int_a^b |f'(x)|dx$

证：$\begin{cases} f(c) - f(a) = \int_a^c f'(x)dx \\ f(b) - f(c) = \int_c^b f'(x)dx \end{cases}$

$\Rightarrow \begin{cases} |f(c)| \le \int_a^c |f'(x)|dx \\ |f(c)| \le \int_c^b |f'(x)|dx \end{cases}$

$2|f(c)| \le \int_a^b |f'(x)|dx$

4. $f'(x) \in C[0,2]$. 证

$|f(2)| \le \left|\frac{1}{2}\int_0^2 f(x)dx\right| + \int_0^2 |f'(x)|dx$

证：$\frac{1}{2}\int_0^2 f(x)dx = f(c) \qquad c \in [0,2]$

$f(2) - f(c) = \int_c^2 f'(x)dx$

$f(2) = f(c) + \int_c^2 f'(x)dx$

$|f(2)| \le |f(c)| + \int_c^2 |f'(x)|dx$

$\le \left|\frac{1}{2}\int_0^2 f(x)dx\right| + \int_0^2 |f'(x)|dx$

Date. /

三. 高阶可导 —— Taylor

$f(x) - T$

$F(x) = \int_a^x f(t)\,dt - T$

① $\begin{cases} \frac{f^{(n)}(\xi)}{n!}(\quad)^n - f(x) \\ \frac{f^{(n+1)}(\xi)}{(n+1)!}(\quad)^{n+1} - F(x) \end{cases}$

② $\begin{cases} \xi \in [a,b] \;.\; f(x) \\ \xi \in (a,b) \;.\; F(x) \end{cases}$

P114. 例1证: 令 $F(x) = \int_2^x f(t)\,dt$. $F'(x) = f(x)$ $F'(3) = 0$.

$$\begin{cases} F(2) = F(3) + \frac{F''(3)}{2!}(2-3)^2 + \frac{F'''(\xi_1)}{3!}(2-3)^3. & 2<\xi_1<3. \\ F(4) = F(3) + \frac{F''(3)}{2!}(4-3)^2 + \frac{F'''(\xi_2)}{3!}(4-3)^3 & 3<\xi_2<4. \end{cases}$$

$$\begin{cases} F(2) = F(3) + \frac{1}{2}F''(3) - \frac{1}{6}f''(\xi_1) \\ F(4) = F(3) + \frac{1}{2}F''(3) + \frac{1}{6}f''(\xi_2) \end{cases}$$

$$\int_2^4 f(x)\,dx = \frac{1}{6}[f''(\xi_1) + f''(\xi_2)]$$

$f''(x) \in C[\xi_1, \xi_2] \Rightarrow \exists\, m, M.$

$$m \le \frac{f''(\xi_1) + f''(\xi_2)}{2} \le M$$

$\exists\, \xi \in [\xi_1, \xi_2] \subset (2,4)$, 使

$$f''(\xi_1) + f''(\xi_2) = 2f(\xi)$$

P114, 例2. 证.

① $f(x) = f(0) + f'(0)x + \frac{f''(\xi)}{2!}x^2 \quad (0<\xi<x)$

② $\int_{-a}^a f(x) = \int_{-a}^a \frac{f''(\xi)}{2!}x^2\,dx$

$f''(x) \in C[-a,a] \Rightarrow \exists\, m, M$

$$\frac{m}{3}a^3 \le \int_{-a}^a \frac{f''(\xi)}{2!}x^2\,dx \le \frac{M}{3}a^3$$

$$m \le \frac{3}{a^3}\int_{-a}^a f(x)\,dx \le M$$

$\exists\, \eta \in [-a,a],\ f''(\eta)$ //

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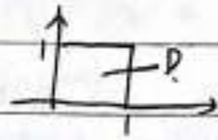
Part Ⅲ 二重积分

一、def：$\iint_D f(x,y)\,d\sigma = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i$

Notes:

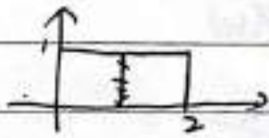
①



$$\lim_{\substack{m\to\infty\\ n\to\infty}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\frac{i}{m},\frac{j}{n}\right)=\iint_D f(x,y)\,d\sigma$$

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f\left(\frac{i}{n},\frac{j}{n}\right)=\iint_D f(x,y)\,d\sigma$$

②



$$\lim_{\substack{m\to\infty\\ n\to\infty}} \frac{1}{mn}\sum_{i=1}^{2m}\sum_{j=1}^{n} f\left(\frac{i}{m},\frac{j}{n}\right)=\iint_D f(x,y)\,d\sigma$$

例1. $\lim_{\substack{m\to\infty\\ n\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{mj}{(m^2+i^2)(n^2+j^2)} = \lim_{\substack{m\to\infty\\ n\to\infty}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\frac{j}{n}}{\left[1+\left(\frac{i}{m}\right)^2\right]\left[1+\left(\frac{j}{n}\right)^2\right]}$

$= \int_0^1 \frac{1}{1+x^2}\,dx \int_0^1 \frac{y}{1+y^2}\,dy$

二、性质：

1.① D左右对称，右D_1，则

If $f(-x,y)=-f(x,y)$，则 $\iint_D f\,d\sigma=0$；

If $f(-x,y)=f(x,y)$，则 $\iint_D f\,d\sigma=2\iint_{D_1} f\,d\sigma$

② D关于$y=x$对称，则

$\iint_D f(x,y)\,d\sigma=\iint_D f(y,x)\,d\sigma$

③ D关于$y=-x$对称，则

$\iint_D f(x,y)\,d\sigma=\iint_D f(-y,-x)\,d\sigma$

2. D—有界闭区域，$f(x,y)\in C(D)$，则

$\exists(\xi,\eta)\in D$，使

$\iint_D f(x,y)\,d\sigma = f(\xi,\eta)\cdot A$.

$\frac{x^2}{t^2}+\frac{y^2}{\frac{t^2}{4}}\le 1$

例2. D：$x^2+4y^2\le t^2\ (t>0)$

求 $\lim_{t\to 0}\frac{\iint_D e^{-x^2}\cos(x+y)\,d\sigma}{t^2}$

解：分子 $= e^{-\xi^2}\cdot\cos(\xi+\eta)\cdot\pi\cdot t\cdot\frac{t}{2}$ $(\xi,\eta)\in D$.

原式 $=\frac{\pi}{2}\lim_{t\to 0} e^{-\xi^2}\cdot\cos(\xi+\eta)=\frac{\pi}{2}$

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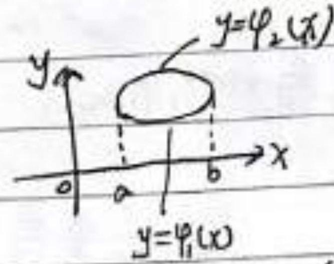
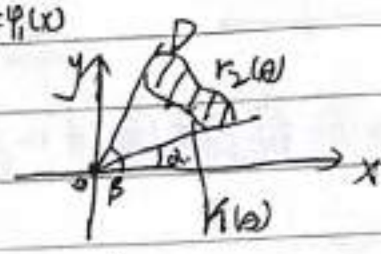
三. 积分法

(一) 直角：



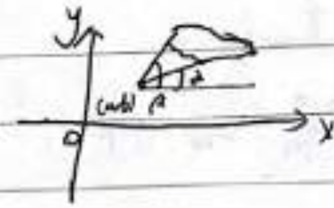
$$\iint f(x,y)\,d\sigma=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy.$$

(二) 极：



1. $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

2. $\begin{cases} x-a=r\cos\theta \\ y-b=r\sin\theta \end{cases}$



3. 如 D: $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$

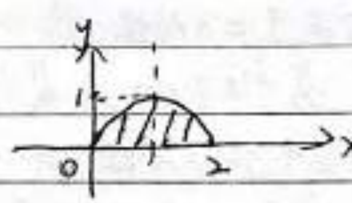
令 $\begin{cases} x=ar\cos\theta \\ y=br\sin\theta \end{cases}$ $(0\le\theta\le 2\pi,\ 0\le r\le 1)$.

$d\sigma=abr\,dr\,d\theta$

型一 改变次序

① 改变次序

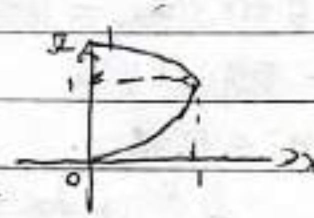
例1. $\int_0^2 dx\int_0^{\sqrt{2x-x^2}} f(x,y)\,dy$ 改次序



解：

原 $=\int_0^1 dy\int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\,dx$

2. $\int_0^1 dx\int_{x^2}^{\sqrt{2-x^2}} f(x,y)\,dy$



解.

原 $=\int_0^1 dy\int_0^{\sqrt{y}} f(x,y)\,dx+\int_1^{\sqrt{2}} dy\int_0^{\sqrt{2-y^2}} f(x,y)\,dx$

② x, y 次序不对，无法计算

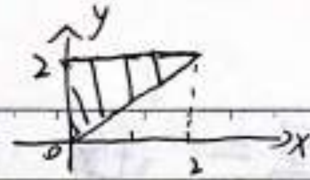
$\begin{cases} x^{2n}e^{\pm x^2}dx \\ e^{\frac{1}{x}}\,dx \\ \cos\frac{1}{x}\,dx,\ \sin\frac{1}{x}\,dx \end{cases}$

例3. $\int_0^1 dx\int_x^{\sqrt{x}} \sin\frac{x}{y}\,dy$

解：

原 $=\int_0^1 dy\int_{y^2}^{y}\sin\frac{x}{y}\,dx=-\int_0^1 y(\cos 1-\cos y)\,dy=-\frac{1}{2}\cos 1+\int_0^1 y\,d(\sin y)=-\frac{1}{2}\cos 1+y\sin y\big|_0^1-\int_0^1 \sin y\,dy$

$=-\frac{1}{2}\cos 1+\sin 1+\cos 1-1$



例4 $\int_0^2 dx\int_x^2 y^2 e^{-y^2}dy$

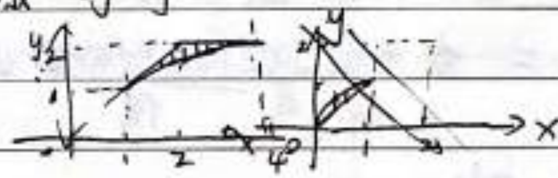
解：

原式 $=\int_0^2 dy\int_0^y y^2e^{-y^2}dx = \int_0^2 y^3e^{-y^2}dy = \frac12\int_0^2 y^2e^{-y^2}d(y^2) = \frac12\int_0^4 te^{-t}dt$

$=-\frac12\int_0^4 t\,d(e^{-t})$

例5. $\int_1^2 dx\int_{\sqrt x}^x \sin\frac{\pi x}{2y}dy + \int_2^4 dx\int_{\sqrt x}^2 \sin\frac{\pi x}{2y}dy$

解：



原式 $=\int_1^2 dy\int_y^{y^2}\sin\frac{\pi x}{2y}dx$

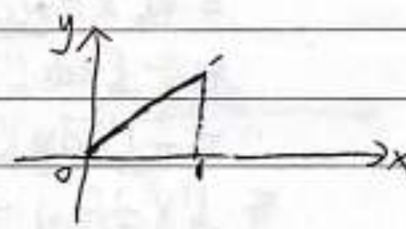
$=-\frac{2}{\pi}\int_1^2 y\cos\frac{\pi}{2}y\,dy = -\frac{8}{\pi^3}\int_1^2 \frac{\pi}{2}y\cos\frac{\pi}{2}y\,d(\frac{\pi}{2}y) = -\frac{8}{\pi^3}\int_{\frac\pi2}^{\pi} t\cos t\,dt$

$=-\frac{8}{\pi^3}\int_{\frac\pi2}^{\pi} t\,d(\sin t) = -\frac{8}{\pi^3}(t\sin t|_{\frac\pi2}^{\pi} - \int_{\frac\pi2}^{\pi}\sin t\,dt)$

$=-\frac{8}{\pi^3}(-\frac{\pi}{2}-1)$

③ 积分次序不对，无法计算

P170. 例6. 解.



原式 $=\int_0^1dx\int_0^x y\sqrt{1-x^2+y^2}\,dy$

$=\frac13\int_0^1[1-(1-x^2)^{\frac32}]\,dx$

$=\frac13-\frac13\int_0^{\frac\pi2}\cos^4t\,dt = \frac13-\frac13\times\frac34\times\frac12\times\frac\pi2$

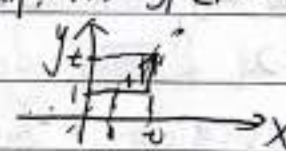
④ 变积分限函数求导

P170 例7

$\varphi(t)=\int_1^t dy\int_y^t f(x)dx$

$\varphi'(t)$?

分析：$\varphi(t)=\int_1^t[F(t)-F(y)]\,dy$



法一：

$\varphi(t)=\int_1^t dx\int_1^x f(x)dy$

$=\int_1^t(x-1)f(x)dx$

$\varphi'(t)=(t-1)f(t)$

法二. 设$F(u)$为$f(u)$的原函数

$\varphi(t)=\int_1^t[F(t)-F(y)]\,dy$

$=(t-1)F(t)-\int_1^tF(y)dy$

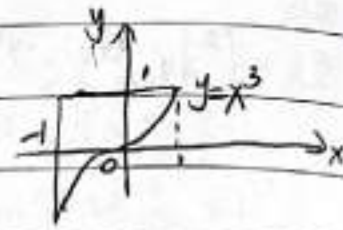
$\varphi'(t)=\cancel{F(t)}+(t-1)f(t)-\cancel{F(t)}$

Date: /

题型二 计算

一、抽象函数计算：

1、$\iint_D x[1+yf(x^2+y^2)]\,d\sigma$，$f(u)$连续



解：原 $=\int_{-1}^{1} x\,dx\int_{x^3}^{1}[1+yf(x^2+y^2)]\,dy$

$=\int_{-1}^{1}x(1-x^3)\,dx+\frac{1}{2}\int_{-1}^{1}x\,dx\int_{x^3}^{1} f(x^2+y^2)\,d(x^2+y^2)$

$=-\frac{2}{5}+\frac{1}{2}\int_{-1}^{1}x[F(x^2+1)-F(x^2+x^6)]\,dx=-\frac{2}{5}$

奇　　偶

2、P174例10

$\iint_{D_{xy}} xyf''_{xy}(x,y)\,d\sigma=\int_0^1 x\,dx\int_0^1 y\,d f'_x(x,y)$

而 $\int_0^1 y\,df'_x(x,y)=yf'_x(x,y)\Big|_0^1-\int_0^1 f'_x(x,y)\,dy$

$=f'_x(x,1)-\int_0^1 f'_x(x,y)\,dy$

原式 $=\int_0^1 x f'_x(x,1)\,dx-\int_0^1 x\,dx\int_0^1 f'_x(x,y)\,dy$

$=\int_0^1 x\,d\underline{f(x,1)}-\int_0^1 x\,dx\int_0^1 f'_x(x,y)\,dy$

$=-\int_0^1 x\,dx\int_0^1 f'_x(x,y)\,dy$

$=-\int_0^1 dy\int_0^1 x\,df(x,y)$

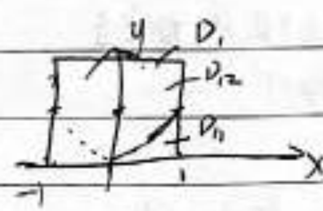
而 $\int_0^1 x\,df(x,y)=xf(x,y)\Big|_0^1-\int_0^1 f(x,y)\,dx$

$=\underline{f(1,y)}-\int_0^1 f(x,y)\,dx$

$\therefore$ 原式 $=\int_0^1 dy\int_0^1 f(x,y)\,dx=a$

二、常规计算

1. $\iint \sqrt{|y-x^2|}\,d\sigma$



解：原 $=2\iint_{D_1}\sqrt{|y-x^2|}\,d\sigma$

$=2\left(\iint_{D_{11}}\sqrt{x^2-y}\,d\sigma+\iint_{D_{12}}\sqrt{y-x^2}\,d\sigma\right)$

$\iint_{D_{11}}\sqrt{x^2-y}\,d\sigma=-\int_0^1 dx\int_0^{x^2}\sqrt{x^2-y}\,d(x^2-y)=-\int_0^1 \frac{2}{3}(x^2-y)^{\frac{3}{2}}\Big|_0^{x^2}dx$

$=-\frac{2}{3}\int_0^1(0-x^3)\,dx=\frac{2}{3}\times\frac{1}{4}=\frac{1}{6}$

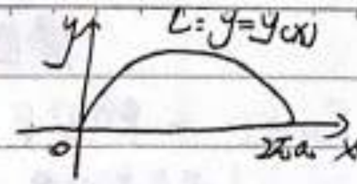
$\iint_{D_{12}}\sqrt{y-x^2}\,d\sigma=\int_0^1 dx\int_{x^2}^{2}(y-x^2)^{\frac{1}{2}}d(y-x^2)=\frac{2}{3}\int_0^1 (y-x^2)^{\frac{3}{2}}\Big|_{x^2}^{2}dx$

$=\frac{2}{3}\int_0^1(2-x^2)^{\frac{3}{2}}dx\ \underline{\underline{x=\sqrt{2}\sin t}}\ \frac{2}{3}\int_0^{\frac{\pi}{4}}2\sqrt{2}\cos^3 t\cdot\sqrt{2}\cos t\,dt$

$=\frac{8}{3}\int_0^{\frac{\pi}{4}}\left(\frac{1+\cos 2t}{2}\right)^2dt=\frac{1}{3}\int_0^{\frac{\pi}{4}}(1+\cos 2t)^2\,d2t=\frac{1}{3}\int_0^{\frac{\pi}{2}}(1+2\cos t+\cos^2 t)\,dt$

$=\frac{1}{3}\left(\frac{\pi}{2}+2\times1+\frac{1}{2}\times\frac{\pi}{2}\right)$

2、P171 例3. 解:



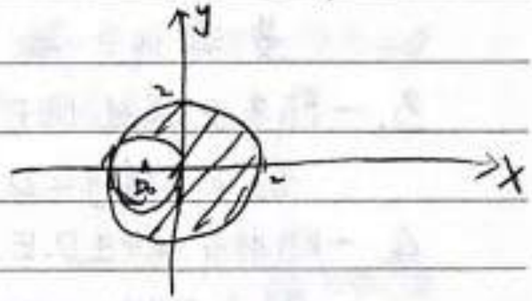
$\iint y\,d\sigma = \int_0^{2\pi a} dx \int_0^{y(x)} y\,dy$

$= \frac{1}{2}\int_0^{2\pi a} y^2(x)\,dx = \frac{1}{2}\int_0^{2\pi a} y^2\,dx$

$= \frac{1}{2}\int_0^{2\pi} a^2(1-\cos t)^2\cdot a(1-\cos t)\,dt = \frac{a^3}{2}\int_0^{2\pi}\left(2\sin^2\frac{t}{2}\right)^3 dt$

$= \frac{a^3}{2}\times 16\int_0^{2\pi}\sin^6\frac{t}{2}\,d\frac{t}{2} = 8a^3\int_0^{\pi}\sin^6 t\,dt = 16a^3\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} =$

3、P172 例5. 解.



原 $= \iint_D \sqrt{x^2+y^2}\,d\sigma$

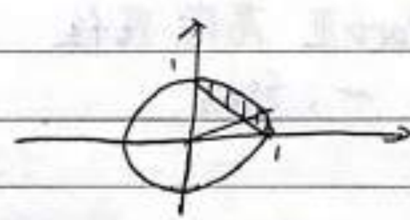
$= \iint_{D+D_0} - \iint_{D_0}$

$\iint_{D+D_0} = \int_0^{2\pi} d\theta\int_0^{2} r^2\,dr = \frac{16}{3}\pi$

$\iint_{D_0}\sqrt{x^2+y^2}\,d\sigma = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta\int_0^{-2\cos\theta} r^2\,dr$

$= -\frac{8}{3}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos^3\theta\,d\theta \overset{\theta-\pi=t}{=} \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3 t\,dt = \frac{16}{3}\times\frac{2}{3}\times 1 = \frac{32}{9}$

4、P172 例6

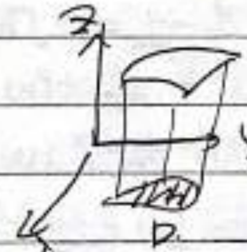


解 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

$\left(0\le\theta\le\frac{\pi}{2},\ \frac{1}{\sin\theta+\cos\theta}\le r\le 1\right)$

原 $I = \int_0^{\frac{\pi}{2}} d\theta\int_{\frac{1}{\sin\theta+\cos\theta}}^{1} dr = \int_0^{\frac{\pi}{2}} 1-\frac{1}{\sin\theta+\cos\theta}\,d\theta$

$= \frac{\pi}{2} - \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\csc\left(\theta+\frac{\pi}{4}\right)d\left(\theta+\frac{\pi}{4}\right)$



$\Sigma: z=\varphi(x,y)$

$A = \iint_D \sqrt{1+z_x'^2+z_y'^2}\,d\sigma$

Date. /

第四模块 微分方程

Part I 一阶D.E

1. 可分离变量

$$\frac{dy}{dx} = \varphi_1(x)\cdot\varphi_2(y) \Rightarrow \int\frac{dy}{\varphi_2(y)} = \int\varphi_1(x)dx + C$$

2. 齐：

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$$

$$\frac{y}{x} \triangleq u \Rightarrow u + x\frac{du}{dx} = \varphi(u) \Rightarrow \int\frac{du}{\varphi(u)-u} = \int\frac{dx}{x} + C$$

3. 一阶齐次线性D.E.

$$\frac{dy}{dx} + P(x)y = 0 \Rightarrow y = Ce^{-\int P(x)dx}$$

4. 一阶非齐次线性D.E.

$$\frac{dy}{dx} + P(x)y = Q(x) \Rightarrow y = \left[\int Q(x)e^{\int P(x)dx}dx + C\right]e^{-\int P(x)dx}$$

Part II 可降阶D.E

Part III 高阶线性

一、结构 $\cdot$ —— $= f(x)$ $\quad f = f_1 + f_2$

$\cdot$ —— $= f_1$

$\cdot$ —— $= f_2$

例1. $y' + P(x)y = Q(x)$

$\Downarrow$

$y_1 = e^x + \sqrt{1+x^2}$,

$y_2 = e^x - \sqrt{1+x^2}$

$P(x),\ Q(x)$?

$y_1 - y_2 = 2\sqrt{1+x^2}$ 为 $y'+P(x)y=0$

$2\cdot\frac{x}{\sqrt{1+x^2}} + P(x)\cdot 2\sqrt{1+x^2} = 0 \Rightarrow P(x) = \ldots$

$y' - \frac{x}{1+x^2}y = Q(x)$

$y_0 = \frac{1}{2}y_1 + \frac{1}{2}y_2 = e^x \Rightarrow Q$

二、特例：

(一) $y'' + py' + qy = 0$.

$\lambda^2 + p\lambda + q = 0$

① $\Delta > 0$, $\lambda_1 \neq \lambda_2$, $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$

② $\Delta = 0$, $\lambda_1 = \lambda_2$ $\quad y = (C_1 + C_2x)e^{\lambda x}$

③ $\Delta < 0$, $\lambda_{1,2} = \alpha \pm i\beta$, $y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$

(二) $y''' + py'' + qy' + ry = 0$

$\lambda^3 + p\lambda^2 + q\lambda + r = 0$

① 如, 1, 2, 3, $\quad y = C_1e^x + C_2e^{2x} + C_3e^{3x}$

② 1, 1, 2, $\quad y = (C_1 + C_2x)e^x + C_3e^{2x}$

③ 1 1 1 $\quad y = (C_1 + C_2x + C_3x^2)e^x$

④ 1, $1\pm 2i$ $\quad y = C_1e^x + e^x(C_2\cos 2x + C_3\sin 2x)$

(三) $y''+py'+qy=f(x)$

型一　defs.

1. $y=e^{2x}+3e^{x}\cos x$ 为 $y'''+py''+qy'+ry=0$ 的特解，求该方程

解：　$\lambda_1=2$　　$\lambda_{2,3}=1\pm i$

$(\lambda-2)(\lambda-1-i)(\lambda-1+i)=0$

$(\lambda-2)(\lambda^2-2\lambda+2)=0$

$\Rightarrow \lambda^3-4\lambda^2+6\lambda-4=0 \Rightarrow y'''-4y''+6y'-4y=0$

2. $y''-2y'=xe^{2x}+x^2+1$. 特解形式 (　　)

(A) $x(ax+b)e^{2x}+Ax^2+bx+c$

(B) $(ax+b)e^{2x}+x(Ax^2+Bx+c)$

(C) $(ax+b)e^{2x}+Ax^2+Bx+c$

√(D) $x(ax+b)e^{2x}+x(Ax^2+Bx+C)$

1. 拆分叠加
2. e^{kx} 的 k.

$\lambda^2-2\lambda=0 \Rightarrow \lambda_1=2,\ \lambda_2=0$

$y''-2y'=xe^{2x}$.　$y_1=x(ax+b)e^{2x}$

$y''-2y'=x^2+1$,　$y_2=x(ax^2+bx+c)$

3. $y=e^{2x}+(1+x)e^{x}$ 为 $y''+ay'+by=ce^{x}$ 的特解. 求 a,b,c.

解：$y=\underset{齐}{e^{2x}}+\underset{齐}{e^{x}}+\underset{特}{xe^{x}}$

代入 解C

$\lambda_1=2$　　$\lambda_2=1$

$(\lambda-2)(\lambda-1)=0 \Rightarrow \lambda^2-3\lambda+2=0$

$a=-3,\ b=2$　　$y''-3y'+2y=ce^{x}$

型二. 解D.E.

1. $xy'+2y=x\ln x$

解：法一：　$y'+\frac{2}{x}y=\ln x$

$y=(\int \ln x\, e^{\int\frac{2}{x}dx}dx+C)\,e^{-\int\frac{2}{x}dx}$

$=(\int x^2\ln x\, dx+C)\,\frac{1}{x^2}$

法二.　$x^2y'+2xy=x^2\ln x$

$(x^2y)'=x^2\ln x$

$x^2y=\int x^2\ln x\, dx+C$

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2、$\frac{dy}{dx}=\frac{1}{x+2y}$

解：$\frac{dx}{dy}=x+2y \Rightarrow \frac{dx}{dy}-x=2y$ 　$x=(\int 2y e^{\int -1 dy}dy+c)e^{-\int -1 dy}$

$=(2\int y e^{-y}dy+c)e^{y}$

3、$2yy'-xy^2=\cancel{e}x$ 求通解.

解：$\frac{d(y^2)}{dx}-xy^2=\cancel{e}x$

$y^2=(\int x e^{\int x dx}dx+c)e^{-\int x dx}$

4、P_{151} 例5 解：$f'(x)=2f(x)$

$f'(x)-2f(x)=0$

$f(x)=ce^{-\int -2dx}=ce^{2x}$

$\because f(0)=\ln 2$ 　$\therefore c=$

5、$f(x)$：$f(x)-4\int_0^x t f(x-t)dt=e^{2x}$ ① 求 $f(x)$

解：$\int_0^x t f(x-t)dt \xlongequal{x-t=u} \int_x^0 (x-u)f(u)(-du)$

$=\int_0^x (x-u)f(u)du=x\int_0^x f(u)du-\int_0^x uf(u)du$

$f'(x)-4\int_0^x f(u)du=2e^{2x}$ ②

$f''(x)-4f(x)=4e^{2x}$ ③

$\lambda^2-4=0 \Rightarrow \lambda_1=-2$ 　$\lambda_2=2.$

$f''(x)-4f(x)=0 \Rightarrow f(x)=C_1e^{-2x}+C_2e^{2x}$

$f_0(x)=axe^{2x}$ 代入③ 　$a=?$

$f(x)=C_1e^{-2x}+C_2e^{2x}+axe^{2x}$

$\because f(0)=1$ 　$f'(0)=2$

P_{152}. 例3 解：　$x=x(y)$：$\frac{d^2x}{dy^2}+(y+\sin x)(\frac{dx}{dy})^3=0$

〈1〉 $x'(y)=\frac{dx}{dy}=\frac{1}{y'(x)}$

$\frac{d^2x}{dy^2}=\frac{d[\frac{1}{y'(x)}]/dx}{dy/dx}=\frac{1}{y'(x)}\cdot -\frac{1}{y'^2(x)}\times y''(x)=-\frac{y''(x)}{y'^3(x)}$ 代入得

$-\frac{y''(x)}{y'^3(x)}+(y+\sin x)\frac{1}{y'^3(x)}=0$

$y''(x)=y+\sin x$ 　即 $\frac{d^2y}{dx^2}-y=\sin x$

(2) 　$\lambda^2-1=0 \Rightarrow \lambda=-1$ ，$\lambda=1$

$\frac{d^2y}{dx^2}-y=0 \Rightarrow y=C_1e^{-x}+C_2e^{x}$

二型 注：
1. 指数函数照抄
2. 按右边构造假设 $\sin x, \cos x$ 都有
3. $\alpha+i\beta$

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~~$y_0(x)=$~~ 如 ① $y''-3y'+2y=2e^x\sin x$ $\quad \alpha=1 \quad \beta=1$

$\lambda_1=1,\ \lambda_2=2.$ $\quad \alpha+i\beta=1+i$

$y_0(x)=e^x(a\cos x+b\sin x)$ ✓

② $y''+4y=x\cos 2x$ $\quad \alpha=0 \quad \beta=2$

$\lambda=\pm 2i$ $\quad \alpha+i\beta=2i$

$y_0(x)=x[(ax+b)\cos 2x+(cx+d)\sin 2x]$

③ $y''-2y'+2y=e^x\cos x$ $\quad \alpha=1,\ \beta=1$

$\lambda=1\pm i.$ $\quad \alpha+i\beta=1+i.$

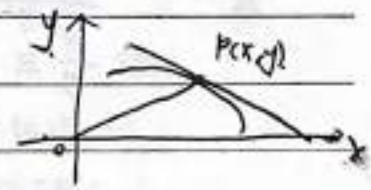
$y_0(x)=xe^x(a\cos x+b\sin x)$

令 $y_0(x)=a\cos x+b\sin x$ 代入 $y''-y=\sin x$

$a=?$, $b=?$

题型三 应用

1. P155 例1. 解：



令 $L: y=y(x,y)$, $P(x,y)\in L.$

切线 $Y-y=y'(X-x)$

令 $Y=0 \Rightarrow X=x+\frac{y}{y'}$

$\therefore \frac{1}{2}\times(x+\frac{y}{y'})+y=k \Rightarrow xy+\frac{y^2}{y'}=2k$

$\Rightarrow \frac{y^2}{y'}=2k-xy \Rightarrow y'=\frac{y^2}{2k-xy}$ 即 $\frac{dy}{dx}=\frac{y^2}{2k-xy}$

$\frac{dx}{dy}=-\frac{1}{y}x+\frac{2k}{y^2}$

$\frac{dx}{dy}+\frac{1}{y}x=\frac{2k}{y^2}$ $\quad x=(\int\frac{2k}{y^2}e^{\int\frac{1}{y}dy}dy+C)e^{-\int\frac{1}{y}dy}$

$=(2k\ln y+C)\frac{1}{y}$

$x=\frac{C}{y}+2k\frac{\ln y}{y}$

$\because x=1$ 时，$y=1$，$\therefore C=1.$

$\therefore x=\frac{1}{y}+\frac{2k\ln y}{y}$

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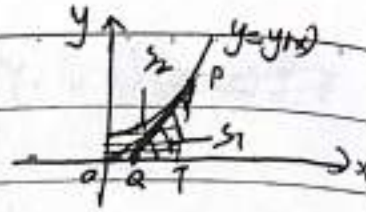
例2. $L: y=y(x)$: $\begin{matrix} y(0)=1 \\ y'(x)>0 \end{matrix}$ $(x\geq 0)$



$2S_1-S_2\equiv 1$ 求 $y(x)$

解: 1° $L: y=y(x)$. $P(x,y)\in L$.

切线 $Y-y=y'(X-x)$

2° 令 $Y=0 \Rightarrow X=x-\frac{y}{y'}$

$S_1=\frac{1}{2}\times\frac{y}{y'}\times y=\frac{y^2}{2y'}$ $\quad S_2=\int_0^x y(t)\,dt$

3° $\frac{y^2}{y'}-\int_0^x y(t)\,dt=1$ ①

4° $\frac{2yy'^2-y^2y''}{y'^2}-y=0 \Rightarrow yy'^2=y^2y''$

$\because \begin{cases} y(0)=1 \\ y'(x)>0 \end{cases}$ $\therefore y(x)\geq 1$ $(x\geq 0)$

$\therefore yy''=y'^2$ 或 $yy''-y'^2=0$.

法一: $\frac{yy''-y'^2}{y^2}=0 \Rightarrow \left(\frac{y'}{y}\right)'=0$

$\Rightarrow \frac{y'}{y}\equiv C_1$.

$\because y(0)=1 \quad \therefore y'(0)=1 \Rightarrow C_1=1 \Rightarrow y'-y=0$

$y=C_2e^{-\int -1\,dx}=C_2e^x$

$\because y(0)=1 \quad \therefore C_2=1 \quad \therefore y=e^x$

可降: $\begin{cases} y^{(n)}=f(x) \\ f(x,y',y'')=0 \quad y'=p \text{ 而 } y''=\frac{dp}{dx} \Rightarrow f(x,p,\frac{dp}{dx})=0 \\ \qquad p=\varphi(x,C_1) \text{ 即 } y=\int\varphi(x,C_1)\,dx+C_2 \\ f(y,y',y'')=0 \quad \text{令 } y'=P, \ \cancel{y''=\frac{dp}{dx} \checkmark \ f(y,P,\frac{dp}{dx})=0.} \\ \qquad y''=\frac{dp}{dx}=\frac{dy}{dx}\frac{dp}{dy}=P\frac{dp}{dy} \\ \qquad f(y,P,P\frac{dp}{dy})=0 \quad P=\varphi(y,C_1) \\ \qquad \text{即 } \frac{dy}{dx}=\varphi(y,C_1) \\ \qquad \Rightarrow \int\frac{dy}{\varphi(y,C_1)}=\int dx+C_2 \end{cases}$

法二：令 $y'=p$ $y''=p\frac{dp}{dy}$ 代入

$y\cdot p\frac{dp}{dy}-p^2=0$

$\because p\neq 0$ $\therefore \frac{dp}{dy}-\frac{1}{y}p=0$ $p=C_1e^{-\int -\frac{1}{y}dy}=C_1y$

即 $y'=C_1y$ $\because y(0)=1$ $\therefore y'(0)=1$ $\therefore C_1=1$

$y'-y=0$

P155 例8 解.

$\frac{\partial u}{\partial x}=f'(v)\cdot\frac{1}{r}\cdot\frac{x}{r}=\frac{x}{r^2}f'(v)$

$\frac{\partial^2 u}{\partial x^2}=\frac{r^2-x\cdot 2r\cdot\frac{x}{r}}{r^4}f'(v)+\frac{x}{r^2}\cdot f''(v)\cdot\frac{1}{r}\cdot\frac{x}{r}$

$=\frac{r^2-2x^2}{r^4}f'(v)+\frac{x^2}{r^4}f''(v)$

$\frac{\partial^2 u}{\partial y^2}=\frac{r^2-2y^2}{r^4}f'(v)+\frac{y^2}{r^4}f''(v)$

$\frac{\partial^2 u}{\partial z^2}=\frac{r^2-2z^2}{r^4}f'(v)+\frac{z^2}{r^4}f''(v)$

$\Rightarrow \frac{1}{r^2}f'(v)+\frac{1}{r^2}f''(v)=\frac{1}{r^2}$

$\Rightarrow f''(v)+f'(v)=e^{-v}$

$\lambda^2+\lambda=0 \Rightarrow \lambda=0,\ \lambda=-1$

$f''(v)+f'(v)=0 \Rightarrow f(v)=C_1+C_2e^{-v}$

$f_0(v)=ave^{-v}$ 代入 $f''(v)+f'(v)=e^{-v} \Rightarrow a=-1.$

$\therefore f(v)=C_1+C_2e^{-v}-ve^{-v}$

代入初始条件.

P156. 例5. 解.

1° 设t时刻半径为 $r(t)$，$r(0)=r_0$ $r(3)=\frac{r_0}{2}$

2° $V(t)=\frac{2}{3}\pi r^3$，$S(t)=2\pi r^2$

3° $\frac{dV}{dt}=\frac{2}{3}\pi\cdot 3r^2\frac{dr}{dt}=2\pi r^2\frac{dr}{dt}$

$\frac{dV}{dt}=-kS \Rightarrow \frac{dR}{dt}=-k$ $r(t)=-kt+C$

$\because r(0)=r_0$ $\therefore C=r_0 \Rightarrow r(t)=-kt+r_0$

$\because r(3)=\frac{r_0}{2}$ $\therefore -3k+r_0=\frac{r_0}{2} \Rightarrow k=\frac{r_0}{6}$

$\therefore r(t)=-\frac{r_0}{6}t+r_0$

4° 令 $r(t)=0 \Rightarrow t=6(h)$

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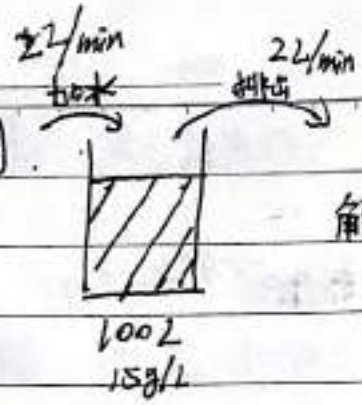
例. 

问几分钟 浓度降半

解: 1° 设t时溶质 $m(t)$, $m(0)=1500$

2° 取 $[t,t+dt]$

$$dm = \lambda - 出 = 0 - \frac{m(t)}{100}\times 2dt$$

$$\Rightarrow \frac{dm}{dt} + \frac{1}{50}m = 0$$

$$m(t) = Ce^{-\int\frac{1}{50}dt} = Ce^{-\frac{1}{50}t}$$

$\because m(0)=1500 \quad \therefore C=1500$

$$\therefore m(t) = 1500e^{-\frac{t}{50}}$$

3° 令 $m(t) = \frac{1}{2}\times 1500$

$$e^{-\frac{t}{50}} = \frac{1}{2} \Rightarrow -\frac{t}{50} = -\ln 2 \quad \therefore t = 50\ln 2\,(min)$$

P156. 例6. 解: 1° 设从2000年初开始 第t年 A含量 $m(t)$ $\quad m(0)=5m_0$.

2°. 取 $[t, t+dt]$

$$dm = \lambda - 出 = \frac{m_0}{V}\times\frac{V}{6}dt - \frac{m(t)}{V}\times\frac{V}{3}dt$$

$$\frac{dm}{dt} = \frac{m_0}{6} - \frac{1}{3}m \qquad \frac{dm}{dt} + \frac{1}{3}m = \frac{m_0}{6}$$

$$m(t) = \left[\int\frac{m_0}{6}e^{\int\frac{1}{3}dt}dt + C\right]e^{-\int\frac{1}{3}dt}$$

$$= \left(\frac{m_0}{2}e^{\frac{t}{3}} + C\right)e^{-\frac{t}{3}} = Ce^{-\frac{t}{3}} + \frac{m_0}{2}$$

由 $m(0)=5m_0 \Rightarrow C = \frac{9}{2}m_0$

$$m(t) = \frac{9}{2}m_0e^{-\frac{t}{3}} + \frac{m_0}{2}$$

3° 令 $m(t)=m_0$

$$\Rightarrow e^{-\frac{t}{3}} = \frac{1}{9} \qquad -\frac{t}{3} = -2\ln 3 \qquad t = 6\ln 3$$

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第五模块 级数

$\begin{cases} \text{常数项级数} \\ \text{幂级数} \\ \text{F级数} \end{cases}$

PartI. 常数项级数

一. defs

1. $\{a_n\}$. $\sum_{n=1}^{\infty} a_n$ ～

2. $S_n = a_1 + a_2 + \cdots + a_n$ —部分和.

① S_n 与 $\sum_{n=1}^{\infty} a_n$ 不同

② $\lim_{n\to\infty} S_n$ 与 $\sum_{n=1}^{\infty} a_n$ 同.

$\lim_{n\to\infty} S_n \begin{cases} = S & \sum_{n=1}^{\infty} a_n = S \\ \text{无.} & \text{发散} \end{cases}$

二. 性质:

1. 收 ± 收 ⇒ 收

2. $\sum_{n=1}^{\infty} a_n$ 与 $\sum_{n=1}^{\infty} k a_n$ $(k\neq 0)$ 敛散性同.

3. 去 加 改 项 级数性不改 可能收敛于别的数

4. 加 (　　) 提高收敛性

如: $\sum_{n=1}^{\infty} a_n$ 收. $\sum_{n=1}^{\infty} (a_{2n-1} + a_{2n})$? $= (a_1 + a_2) + (a_3 + a_4) + \cdots$ 收敛✓

5. $\sum_{n=1}^{\infty} a_n$ 收敛 $\underset{\not\Leftarrow}{\Rightarrow}$ $\lim_{n\to\infty} a_n = 0$

证: "⇒" $S_n = a_1 + \cdots + a_n$

$\lim_{n\to\infty} S_n = S$　　$a_n = S_n - S_{n-1}$　　$\lim_{n\to\infty} a_n = S - S = 0$

"$\not\Leftarrow$"　$\sum_{n=1}^{\infty} \frac{1}{n}$ 发散　　$\frac{1}{n} \to 0\ (n\to\infty)$

例1. $\sum_{n=1}^{\infty} a_n$ 收敛　证: $\sum_{n=1}^{\infty} (a_n + a_{n+1})$ 收敛.

证: $S_n = a_1 + \cdots a_n$.

$\sum_{n=1}^{\infty} a_n$ 收敛 ⇒ $\begin{cases} \lim_{n\to\infty} S_n = S \\ \lim_{n\to\infty} a_n = 0 \end{cases}$

$S_n^{(1)} = (a_1 + a_2) + (a_2 + a_3) + \cdots + (a_n + a_{n+1})$

$= 2S_n - a_1 + a_{n+1}$

$\because \lim_{n\to\infty} S_n^{(1)} = 2S - a_1$

$\therefore \sum_{n=1}^{\infty} (a_n + a_{n+1})$ 收敛.

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例2. $\sum_{n=1}^{\infty} a_n$ 收敛，$\sum_{n=1}^{\infty} b_n$ 绝对收敛

证：$\sum_{n=1}^{\infty} a_n b_n$ 绝对收敛.

证：$\sum_{n=1}^{\infty} a_n$ 收敛 $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

$\exists M>0$，使 $|a_n| \le M$

$0 \le |a_n b_n| \le M|b_n|$

$\because \sum_{n=1}^{\infty} M|b_n|$ 收敛

$\therefore \sum_{n=1}^{\infty} |a_n b_n|$ 收敛.

常：正、交、任意

二、正项级数

(一) $\sum_{n=1}^{\infty} a_n \quad (a_n \ge 0)$

(二) 判别法：

Th1.（比较法）$a_n \ge 0$，$b_n \ge 0$

① $a_n \le b_n$ 且 $\sum_{n=1}^{\infty} b_n$ 收敛 $\Rightarrow \sum_{n=1}^{\infty} a_n$ 收敛

② $a_n \ge b_n$ 且 $\sum_{n=1}^{\infty} b_n$ 发散 $\Rightarrow \sum_{n=1}^{\infty} a_n$ 发散

Th1′ $a_n \ge 0$，$b_n \ge 0$

$\lim\limits_{n\to\infty} \frac{b_n}{a_n} = k \ (\ne 0, \infty)$ 敛散性同

Th1″ $a_n \ge 0$，$b_n \ge 0$

√ ① If $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = 0$ 且 $\sum_{n=1}^{\infty} a_n$ 收敛 $\Rightarrow \sum_{n=1}^{\infty} b_n$ 收敛

② If $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = \infty$ 且 $\sum_{n=1}^{\infty} a_n$ 发散 $\Rightarrow \sum_{n=1}^{\infty} b_n$ 发散

① 证明：取 $\varepsilon = 1$

$\because \lim\limits_{n\to\infty} \frac{b_n}{a_n} = 0$，$\therefore \exists N>0$ 当 $n>N$ 时

$\left|\frac{b_n}{a_n} - 0\right| < 1 \Rightarrow 0 \le b_n < a_n$

$\because \sum_{n=1}^{\infty} a_n$ 收敛. $\therefore \sum_{n=1}^{\infty} b_n$ 收敛.

② $\because \lim\limits_{n\to\infty} \frac{b_n}{a_n} = +\infty \quad \therefore \forall M>0 \ \exists N>0$，当 $n>N$ 时，

$\frac{b_n}{a_n} > M \Rightarrow 0 \le M a_n < b_n$

$\because \sum_{n=1}^{\infty} M a_n$ 发散 $\quad \therefore \sum_{n=1}^{\infty} b_n$ 发散

补：Th4.（积分法）$\sum_{n=1}^{\infty} a_n \ (a_n \ge 0)$ 令 $a_n = f(n)$

If $\{a_n\}\downarrow$，则 $\sum_{n=1}^{\infty} a_n$ 敛散性与 $\int_1^{+\infty} f(x)\,dx$ 敛散性同

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例3. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$?

解: $\{\frac{1}{n\ln n}\}\downarrow$.

$\because \int_2^{+\infty}\frac{1}{x\ln x}dx = \ln\ln x\Big|_2^{+\infty} = +\infty$

$\therefore \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ 发散.

例4. $\sum_{n=2}^{\infty}\frac{1}{n\ln^2 n}$?

解: $\frac{1}{n\ln^2 n}>0$

$\because \{\frac{1}{n\ln^2 n}\}\downarrow$ 且 $\int_2^{+\infty}\frac{1}{x\ln^2 x}dx = -\frac{1}{\ln x}\Big|_2^{+\infty} = \frac{1}{\ln 2}$

$\therefore \sum_{n=2}^{\infty}\frac{1}{n\ln^2 n}$ 收敛.

四 交错:

(一) def —
$$\begin{cases}\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1-a_2+a_3-a_4+\cdots \\ \sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1+a_2-a_3+a_4-\cdots\end{cases}$$
$(a_n>0)$

(二) 判.

Th. $\sum_{n=1}^{\infty}(-1)^{n-1}a_n \quad (a_n>0)$:

If ① $\{a_n\}\downarrow$; ② $\lim_{n\to\infty}a_n=0 \Rightarrow$ 收敛

Q1. $\sum_{n=1}^{\infty}(-1)^{n}a_n\ (a_n>0)$ 仅满足② $\overset{?}{\Rightarrow}$ 收敛. X

反例. $a_n=\frac{1}{n}+(-1)^{n-1}\sin\frac{1}{n}$

$a_n>0$ 且 $\lim_{n\to\infty}a_n=0$

$\sum_{n=1}^{\infty}(-1)^{n-1}a_n=\sum_{n=1}^{\infty}[\frac{(-1)^{n-1}}{n}+\sin\frac{1}{n}]$

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ 收敛. $(\because\{\frac{1}{n}\}\downarrow$ 且 $\frac{1}{n}\to 0\ (n\to\infty))$

$\because \sin\frac{1}{n}\sim\frac{1}{n}$ 且 $\sum_{n=1}^{\infty}\frac{1}{n}$ 发散 $\therefore\sum_{n=1}^{\infty}\sin\frac{1}{n}$ 发散.

$\therefore \sum_{n=1}^{\infty}(-1)^{n-1}a_n$ 发散

Q 2. $\sum_{n=1}^{\infty}a_n$ 收敛 $\overset{?}{\Longrightarrow}\sum_{n=1}^{\infty}a_n^2$? X 不一定

反例. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ 收敛 ($\because\{\frac{1}{\sqrt{n}}\}\downarrow$ 且 $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$)

而 $\sum_{n=1}^{\infty}[\frac{(-1)^n}{\sqrt{n}}]^2$ 发散

Q 3 $\sum_{n=1}^{\infty}a_n\ (a_n\geq 0)$ 收敛 $\overset{?}{\Longrightarrow}\sum_{n=1}^{\infty}a_n^2$ 收 √.

证: $\sum_{n=1}^{\infty}a_n$ 收 $\Rightarrow \lim_{n\to\infty}a_n=0$

取 $\varepsilon=1$ $\exists N>0$, 当 $n>N$ 时 $|a_n-0|<1 \Rightarrow 0\leq a_n<1$

$\Rightarrow 0\leq a_n^2\leq a_n<1$ $\because\sum_{n=1}^{\infty}a_n$ 收敛 $\therefore\sum_{n=1}^{\infty}a_n^2$ 收敛.

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1、P198.例1.证：

(1) $a_n+a_{n+2}=\int_0^{\frac{\pi}{4}}\tan^n x\,d(\tan x)=\frac{1}{n+1}\tan^{n+1}x\Big|_0^{\frac{\pi}{4}}=\frac{1}{n+1}$

原式$=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+1})$

$S_n=(1-\frac{1}{2})+\cdots\cdots+(\frac{1}{n}-\frac{1}{n+1})=1-\frac{1}{n+1}$

$\because \lim_{n\to\infty}S_n=1$ $\therefore$ 原$=1$

(2) $a_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx \xlongequal{\tan x=t}\int_0^1 t^n\frac{1}{1+t^2}dt\le\int_0^1 t^n dt=\frac{1}{n+1}\le\frac{1}{n}$

$0<\frac{a_n}{n^\lambda}\le\frac{1}{n^{\lambda+1}}$

$\because \lambda>0$ $\therefore \sum_{n=1}^{\infty}\frac{1}{n^{\lambda+1}}$ 收敛.

$\therefore \sum_{n=1}^{\infty}\frac{a_n}{n^\lambda}$ 收敛

例2，$a_n>0$. $\sum_{n=1}^{\infty}a_n$发散，$S_n=a_1+\cdots+a_n$，证$\sum_{n=2}^{\infty}\frac{a_n}{S_n}$ 收敛

证：$\because \{S_n\}\uparrow$ $\therefore \lim_{n\to\infty}S_n=+\infty$

$0<\frac{a_n}{S_n}\le\frac{a_n}{S_{n-1}S_n}=\frac{S_n-S_{n-1}}{S_{n-1}S_n}=\frac{1}{S_{n-1}}-\frac{1}{S_n}$

对$\sum_{n=2}^{\infty}(\frac{1}{S_{n-1}}-\frac{1}{S_n})$ $S_n^{(1)}=(\frac{1}{S_1}-\frac{1}{S_2})+(\frac{1}{S_2}-\frac{1}{S_3})+\cdots+(\frac{1}{S_{n-1}}-\frac{1}{S_n})=\frac{1}{S_1}-\frac{1}{S_n}$

$\because \lim_{n\to\infty}S_n^{(1)}=\frac{1}{S_1}$

$\therefore \sum_{n=2}^{\infty}(\frac{1}{S_{n-1}}-\frac{1}{S_n})$ 收敛

3.P198 例2，证：

(1) $f(x)=x^n+nx-1$ $f(0)=-1<0$. $f(1)=n>0$

$\exists x_n\in(0,1)$，$f(x_n)=0$.

又 $f'(x)=nx^{n-1}+n>0$ $(x>0)$

$\Rightarrow f(x)$在$(0,+\infty)\uparrow$，$\therefore x_n$唯一.

(2) $x_n^n+nx_n-1=0$

$0<x_n=\frac{1}{n}(1-x_n^n)\le\frac{1}{n}$

$0<x_n^\alpha\le\frac{1}{n^\alpha}$

$\because \alpha>1$ $\therefore \sum_{n=1}^{\infty}\frac{1}{n^\alpha}$ 收敛

4. P198 例3.证：

(1) $a_{n+1}\ge1$ $a_{n+1}-a_n=\frac{1}{2}(a_n+\frac{1}{a_n})-a_n=\frac{1-a_n^2}{2a_n}\le0$

$\Rightarrow\{a_n\}\downarrow\Rightarrow\lim_{n\to\infty}a_n\exists$.

(2) ∵ $a_n>0$ 且 $\{a_n\}\downarrow$ ∴ $\frac{a_n}{a_{n+1}}-1\geq 0$

$0\leq\frac{a_n}{a_{n+1}}-1=\frac{a_n-a_{n+1}}{a_{n+1}}\leq a_n-a_{n+1}$

对 $\sum_{n=1}^{\infty}(a_n-a_{n+1})$ $S_n=(a_1-a_2)+\cdots+(a_n-a_{n+1})=2-a_{n+1}$

∵ $\lim_{n\to\infty}S_n$ ∃ ∴ $\sum_{n=1}^{\infty}(a_n-a_{n+1})$ 收敛 ∴ $\sum_{n=1}^{\infty}(\frac{a_n}{a_{n+1}}-1)$ 收敛

5. P149 例14. 证 $\frac{a_{n+1}}{a_n}\leq\frac{b_{n+1}}{b_n}$

~~即~~ $\frac{a_{n+1}}{b_{n+1}}\leq\frac{a_n}{b_n}\Rightarrow\{\frac{a_n}{b_n}\}\downarrow$

∵ $\frac{a_n}{b_n}>0$ ∴ $\lim_{n\to\infty}\frac{a_n}{b_n}=A\geq 0$

Case1. $A>0$, $\sum_{n=1}^{\infty}a_n$ 与 $\sum_{n=1}^{\infty}b_n$ 敛散性同 (1).(2)成立.

Case2. $A=0$, 即 $\lim_{n\to\infty}\frac{a_n}{b_n}=0$ 若 $\sum_{n=1}^{\infty}b_n$ 收敛, $\sum_{n=1}^{\infty}a_n$ 收敛. …

例. ① $\{a_n\}\downarrow$; ② $a_n>0$; ③ $\sum_{n=1}^{\infty}(-1)^n a_n$ 发散.

问 $\sum_{n=1}^{\infty}(\frac{1}{1+a_n})^n$?

解: $\{a_n\}\downarrow$ 且 $a_n>0\Rightarrow\lim_{n\to\infty}a_n$ ∃.

令 $\lim_{n\to\infty}a_n=A\geq 0$ $\sum_{n=1}^{\infty}(-1)^n a_n$ 发散 $\Rightarrow A\neq 0\Rightarrow A>0$

$\lim_{n\to\infty}\sqrt[n]{(\frac{1}{1+a_n})^n}=\lim_{n\to\infty}\frac{1}{1+a_n}=\frac{1}{1+A}=\rho<1$ 收敛.

Part II 幂级数

$\sum_{n=0}^{\infty}a_nx^n$ R 1° $\begin{cases}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\rho\\ \lim_{n\to\infty}\sqrt[n]{|a_n|}=\rho\end{cases}\Rightarrow R=\frac{1}{\rho}$

Notes: ① $\sum_{n=0}^{\infty}a_nx^{2n+1}$, $R=\sqrt{\frac{1}{\rho}}$

② $x\in(-R,R)$. 级数绝对收敛.

$x\in(-\infty,-R)\cup(R,+\infty)$. 发散

$x=-R$, $x=R$ 一切皆可能

If $\sum_{n=0}^{\infty}a_nx^n$ 在 $x=x_0$ 条件收敛, 则 $R=|x_0|$

例1. $\sum_{n=0}^{\infty}a_n(2x-1)^n$ 在 $x=-2$ 收敛 在 $x=3$ 发散, $R=$ ____

解: $|2\times(-2)-1|\leq R\Rightarrow R\geq 5$ $|2\times 3-1|\geq R\Rightarrow R\leq 5\Rightarrow R=5$

2° 工具一:

① $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ $(-\infty<x<+\infty)$ ② $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$ $(-\infty<x<+\infty)$

③ $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ $(-\infty<x<+\infty)$ ④ $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ $(-1<x<1)$

⑤ $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^nx^n$ $(-1<x<1)$ ⑥ $\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$ $(-1<x\leq 1)$

⑧ $\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$ ⑦ $-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$ $(-1\leq x<1)$

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注：$\ln 2=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$

工具二：

对 $\sum_{n=0}^{\infty}a_nx^n$：当 $x\in(-R,R)$ 时

Th1. $\left(\sum_{n=0}^{\infty}a_nx^n\right)'=\sum_{n=1}^{\infty}na_nx^{n-1}$ ；　　第一项为常数，求导后没了

Th2. $\int_0^x\left(\sum_{n=0}^{\infty}a_nx^n\right)dx=\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$

任务：

一、展成幂级数

1. 把 $f(x)=\frac{1}{(1+x)^2}$ 展成 x 的幂级数.

解：$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^nx^n \quad (-1<x<1)$

$-\frac{1}{(1+x)^2}=\sum_{n=1}^{\infty}(-1)^nnx^{n-1}$

$f(x)=\sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1} \quad (-1<x<1)$　　端点可能有差异

2. $f(x)=\frac{1}{x^2-1}$ 展成 $x-2$ 的幂级数.

解：$f(x)=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$

$\frac{1}{x-1}=\frac{1}{1+(x-2)}=\sum_{n=0}^{\infty}(-1)^n(x-2)^n \quad (1<x<3)$

$\frac{1}{x+1}=\frac{1}{3+(x-2)}=\frac{1}{3}\cdot\frac{1}{1+\frac{x-2}{3}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-2)^n \quad (-1<x<5)$

$\therefore f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2}\left(1-\frac{1}{3^{n+1}}\right)(x-2)^n \quad (1<x<3)$.

3. $f(x)=\frac{5x-1}{x^2-x-2}$ 展成 $(x+1)$ 的幂级数

解：$f(x)=\frac{5x-1}{(x+1)(x-2)}=2\cdot\frac{1}{x+1}+3\cdot\frac{1}{x-2}$

$\frac{1}{x+1}=\frac{1}{2+(x-1)}=\frac{1}{2}\cdot\frac{1}{1+\frac{x-1}{2}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-1)^n \quad \left(\cancel{0<x<2}\right)\ -1<x<3$

$\frac{1}{x-2}=\frac{1}{-1+(x-1)}=-\frac{1}{1-(x-1)}=-\sum_{n=0}^{\infty}(x-1)^n \quad (0<x<2)$

$f(x)=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^n}-3\right](x-1)^n \quad (0<x<2)$

4. $f(x)=\arctan\frac{1+x}{1-x}$ 展成 x 的幂级数.

解：$f(0)=\frac{\pi}{4}$

$f'(x)=\frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\left(\frac{1+x}{1-x}\right)'=\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^nx^{2n} \quad (-1<x<1)$

$f(x)-f(0)=\int_0^xf'(x)dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$

$f(x)=\frac{\pi}{4}+\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \quad (-1\le x<1)$　　在1处无定义

二. 求$S(x)$

工具 $\begin{cases} ① \ 1)\sim 7) \\ ② \ Th1,\ Th2 \\ ③ \ D.E. \end{cases}$

Case1. $\sum P(n)x^n$

例1. $\sum_{n=0}^{\infty} n(2n+1)x^n$, 求$S(x)$.

解: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1 \Rightarrow R=1$

$x=\pm1$时, ∵ $n(2n+1)(\pm1)^n \not\to 0\ (n\to\infty)$

∴ $x=\pm1$时发散. ∴收敛域 $x\in(-1,1)$.

$S(x)=2\sum_{n=1}^{\infty} n^2x^n+\sum_{n=1}^{\infty} nx^n = 2\sum_{n=1}^{\infty}[n(n-1)+n]x^n+\sum_{n=1}^{\infty} nx^n$

$= 2x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} + 3x\sum_{n=1}^{\infty} nx^{n-1} = 2x^2\left(\sum_{n=2}^{\infty} x^n\right)''+3x\left(\sum_{n=1}^{\infty} x^n\right)'$

$=2x^2\left(\frac{x^2}{1-x}\right)''+3x\left(\frac{x}{1-x}\right)'$

Case2. $\sum \frac{x^n}{P(n)}$ $\begin{cases} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n=\ln(1+x),\ \sum_{n=1}^{\infty}\frac{x^n}{n}=-\ln(1-x) \\ \text{去掉分母.} \end{cases}$

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例3. 解: 1° $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1\Rightarrow R=1$

$x=\pm1$时 $\left|\frac{(\pm1)^n}{n(n+1)}\right|\sim\frac{1}{n^2}$ $\Rightarrow x=\pm1$时收敛.

∴收敛域 $[-1,1]$.

2° $S(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}-\sum_{n=1}^{\infty}\frac{x^n}{n+1}$

① $S(0)=0$;

② $x\neq0$时, ~~$S(x)$~~

$\sum_{n=1}^{\infty} \frac{x^n}{n}=-\ln(1-x)$ $(-1\le x<1)$ 且$x\neq0$.

$\sum_{n=1}^{\infty}\frac{x^n}{n+1}=\frac1x\sum_{n=2}^{\infty}\frac{x^n}{n}=\frac1x\left(\sum_{n=1}^{\infty}\frac{x^n}{n}-x\right)=-\frac1x[\ln(1-x)+x]$ $(-1\le x<1$且$x\neq0)$

$S(x)=\left(\frac1x-1\right)\ln(1-x)+1$. $(-1\le x<1$且$x\neq0)$

③ $S(1)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$

$$S(x)=\begin{cases} 0, & x=0 \\ 1, & x=1 \\ \left(\frac1x-1\right)\ln(1-x)+1, & -1\le x<1 \text{ 且 } x\neq0 \end{cases}$$

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P224. 例4. 1° $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1 \Rightarrow R=1$

$x=\pm1$时，$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$ 收敛，收敛域$[-1,1]$

$2°, S(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}x^{2n}=x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}x^{2n-1}=x\int_0^x\left(\sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2}\right)dx$

$=x\int_0^x\frac{1}{1+x^2}dx=x\arctan x$

P224 例5. 解. 1° $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1 \Rightarrow R=1$；

$x=\pm1$时 $\frac{4n^2+4n+3}{2n+1}\not\to0\ (n\to\infty)$

$x=\pm1$时 发散．收敛域$(-1,1)$

$2°\ S(x)=\sum_{n=0}^{\infty}\frac{4n^2+4n+3}{2n+1}x^{2n}=\sum_{n=0}^{\infty}(2n+1)x^{2n}+2\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1}$

① $S(0)=3$；

② $x\neq0$时．

$\sum_{n=0}^{\infty}(2n+1)x^{2n}=\left(\sum_{n=0}^{\infty}x^{2n+1}\right)'=\left(\frac{x}{1-x^2}\right)'$

$\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1}=\frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}=\frac{1}{x}\int_0^x\left(\sum_{n=0}^{\infty}x^{2n}\right)dx$

$=\frac{1}{x}\int_0^x\frac{1}{1-x^2}dx=-\frac{1}{x}\int_0^x\frac{1}{x^2-1}dx=-\frac{1}{2x}\ln\left|\frac{x-1}{x+1}\right|=-\frac{1}{2x}\ln\frac{1-x}{1+x}$

例 求$\sum_{n=1}^{\infty}\frac{x^{2n}}{n(2n-1)}$ 的$S(x)$

解：1°，$R=1$　$x=\pm1$时 $\frac{(\pm1)^{2n}}{n(2n-1)}\sim\frac{1}{2n^2}\Rightarrow x=\pm1$时收敛

∴收敛域$[-1,1]$．

$2°\ S(x)=\sum_{n=1}^{\infty}\frac{x^{2n}}{n(2n-1)}=2\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n-1)\cdot2n}=2\left(\sum_{n=1}^{\infty}\frac{x^{2n}}{2n-1}-\sum_{n=1}^{\infty}\frac{x^{2n}}{2n}\right)$

$\sum_{n=1}^{\infty}\frac{x^{2n}}{2n-1}=x\int_0^x\left(\sum_{n=1}^{\infty}x^{2n-2}\right)dx=x\int_0^x\frac{1}{1-x^2}dx=-\frac{x}{2}\ln\frac{1-x}{1+x}\quad(-1<x<1)$

$\sum_{n=1}^{\infty}\frac{x^{2n}}{2n}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(x^2)^n}{n}=-\frac{1}{2}\ln(1-x^2)\quad(-1<x<1)$

$S(x)=\ln(1-x^2)-x\ln\frac{1-x}{1+x}\quad(-1<x<1)$

$S(\pm1)=\sum_{n=1}^{\infty}\frac{1}{n(2n-1)}=2\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)=2\ln2$

Case 3. $\sum\frac{x^n}{?!}$ $\begin{cases}e^x.\ \sin x.\ \cos x.\\ D.E.\end{cases}$

1、$\sum_{n=0}^{\infty}\frac{n^2+1}{n!}=?$

解：$S(x)=\sum_{n=0}^{\infty}\frac{n^2+1}{n!}x^n$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0\Rightarrow R=+\infty\Rightarrow x\in(-\infty,+\infty)$

$S(x)=\sum_{n=1}^{\infty}\frac{n^2}{n!}x^n+\sum_{n=0}^{\infty}\frac{x^n}{n!}=\sum_{n=1}^{\infty}\frac{(n-1)+1}{(n-1)!}x^n+e^x$

$=\sum_{n=2}^{\infty}\frac{(n-1)}{(n-1)!}x^n+\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}+e^x=x^2\sum_{n=2}^{\infty}\frac{x^{n-2}}{(n-2)!}+x\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}+e^x$

$=x^2e^x+xe^x+e^x$　　$\sum_{n=0}^{\infty}\frac{n^2+1}{n!}=3e$

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P_{205} 例6，解：

$$y(x)=\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n)!}=1+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\cdots$$

(1) $y'=\frac{x^2}{2!}+\frac{x^5}{5!}+\frac{x^8}{8!}+\cdots$

$y''=x+\frac{x^4}{4!}+\frac{x^7}{7!}+\cdots$

$y''+y'+y=e^x$

(2) $\lambda^2+\lambda+1=0 \Rightarrow \lambda=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$.

$y=e^{-\frac{x}{2}}(C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x)+\frac{1}{3}e^x$

$\because y(0)=1,\ y'(0)=0$

$\therefore$

P_{205} 例7.解：

$y(x)=\sum_{n=0}^{\infty}a_nx^n,\ y(0)=0.\ y'(0)=1 \Rightarrow a_0=0.\ a_1=1.$

$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}\left(=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right)$

$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$

$y''-2xy'-4y$

$=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n-2\sum_{n=1}^{\infty}na_nx^n-4\sum_{n=0}^{\infty}a_nx^n$

$=2a_2-4a_0+\sum_{n=1}^{\infty}[(n+1)(n+2)a_{n+2}-2(n+2)a_n]x^n=0$

$\Rightarrow a_2=0,\quad a_{n+2}=\frac{2}{n+1}a_n$

$a_0=0,\ a_2=0,\ a_4=0,\ \cdots$ 即 $a_{2k}=0\ (k=0,1,2\cdots)$

$a_1=1,\ a_3=\frac{2}{2},\ a_5=\frac{2}{4}\cdot\frac{2}{2},\ a_7=\frac{2}{6}\cdot\frac{2}{4}\cdot\frac{2}{2}$

$=\frac{1}{2}\times\frac{1}{1}\qquad=\frac{1}{3}\times\frac{1}{2}\times\frac{1}{1}=\frac{1}{3!}$

$a_{2k}=0\ (k=0,1,2,\cdots)$

$a_1=1\quad a_3=\frac{1}{1!}\quad a_5=\frac{1}{2!}\quad a_7=\frac{1}{3!}$

$y(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots$

$=x+\frac{x^3}{1!}+\frac{x^5}{2!}+\frac{x^7}{3!}+\cdots$

$=x[1+\frac{(x^2)^1}{1!}+\frac{(x^2)^2}{2!}+\frac{(x^2)^3}{3!}+\cdots]$

$=xe^{x^2}$

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专题五：

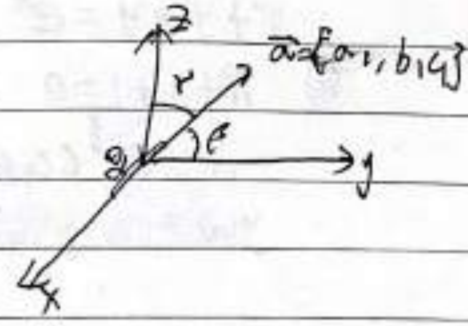
一、空间解析几何

Part I 工具 — 向量

(一) defs:

1. 向量

2、方向角，方向余弦 —



$\vec{a}=\{a_1,b_1,c_1\}$

$|\vec{a}|=\sqrt{a_1^2+b_1^2+c_1^2}$

$\vec{a}^{\circ}=\frac{1}{|\vec{a}|}\vec{a}$

$\begin{cases}\cos\alpha=\frac{a_1}{|\vec{a}|}\\ \cos\beta=\frac{b_1}{|\vec{a}|}\\ \cos\gamma=\frac{c_1}{|\vec{a}|}\end{cases}$ $\begin{cases}① \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1\\ ② \{\cos\alpha,\cos\beta,\cos\gamma\}=\vec{a}^{\circ}\end{cases}$

(二) 向量运算：

1. 几何

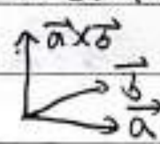
2、代数：

① $\vec{a}\pm\vec{b}=$

② $k\vec{a}=$

③ $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta=a_1a_2+b_1b_2+c_1c_2$

④ $\vec{a}\times\vec{b}$：



几何：$\begin{cases}\text{方向：右手}\\ \text{大小：}|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta\end{cases}$

代数：

$\vec{a}\times\vec{b}=\{\quad,\quad,\quad\}$

$$\begin{matrix}a_1 & b_1 & c_1 & a_1 & b_1 & c_1\\ a_2 & b_2 & c_2 & a_2 & b_2 & c_2\end{matrix}$$

如：$\{1,-1,2\}\times\{3,1,4\}=\{-6,2,4\}$

$$\begin{matrix}1 & -1 & 2 & 1 & -1 & 2\\ 3 & 1 & 4 & 3 & 1 & 4\end{matrix}$$

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Notes:

1. $\vec{a}\cdot\vec{b}$:

① $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

② $\vec{a}\cdot\vec{a}=|\vec{a}|^2$, $\vec{a}\cdot\vec{a}=0 \Leftrightarrow \vec{a}=\vec{0}$

③ $\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$

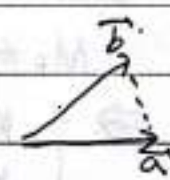
2. $\vec{a}\times\vec{b}$:

① $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$;

② $\vec{a}\times\vec{b}\perp\vec{a}$, $\vec{a}\times\vec{b}\perp\vec{b}$.

③ $\vec{a}\times\vec{b}=\vec{0} \Leftrightarrow \vec{a}\parallel\vec{b}$

④ $|\vec{a}\times\vec{b}|=2S_{\triangle}$



二、应用.

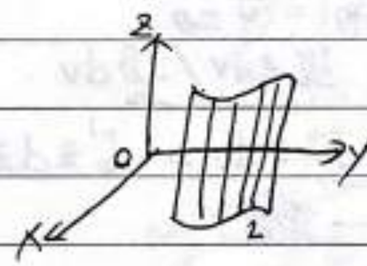
(一) 空间曲面

$$\Sigma: F(x,y,z)=0$$

1. 特殊曲面:

① 柱面:

$\Sigma: F(x,y)=0$



$L: \begin{cases} F(x,y)=0 \\ z=0 \end{cases}$

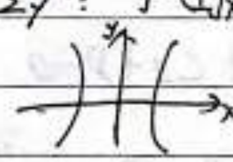
② 旋转曲面

Case 1: 2-dim

$L: \begin{cases} f(x,y)=0 \\ z=0 \end{cases}$ $\Sigma_{x轴}: f(x,\pm\sqrt{y^2+z^2})=0$

$\Sigma_y: f(\pm\sqrt{x^2+z^2}, y)=0$

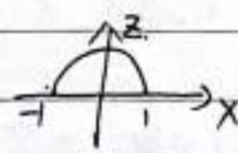
如: $L: \begin{cases} \frac{x^2}{4}-\frac{y^2}{3}=1 \\ z=0 \end{cases}$



$\Sigma_x: \frac{x^2}{4}-\frac{y^2}{3}-\frac{z^2}{3}=1$.

$\Sigma_y: \frac{x^2}{4}-\frac{y^2}{3}+\frac{z^2}{4}=1$

又如: $L: \begin{cases} z=1-x^2 \\ y=0 \end{cases}$



$\Sigma_z: z=1-x^2-y^2$.

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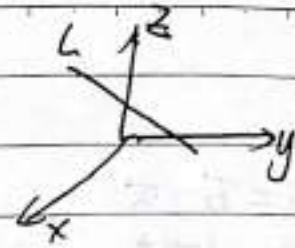
Case 2. 3-dim.



例1. $L: \frac{x-2}{1}=\frac{y+1}{2}=\frac{z}{1}$

① Σ_z;

② Σ_z 介于 $z=0$ 与 $z=1$ 之间 V.

③ 质心坐标.

解: ① $\forall M(x,y,z)\in\Sigma_z$. $M_0(x_0,y_0,z)\in L$.

轨迹法 $T(0,0,z)$

找等量关系 由 $|MT|=|M_0T| \Rightarrow x^2+y^2=x_0^2+y_0^2$

$\because M_0\in L$. $\therefore \frac{x_0-2}{1}=\frac{y_0+1}{2}=\frac{z}{1}$

$$\Rightarrow \begin{cases} x_0=2+z \\ y_0=-1+2z \end{cases}$$

Σ_z: $x^2+y^2=5(1+z^2)$

② $V=\iiint_\Omega 1\,dV=\int_0^1 dz\iint_{D_z} dxdy = 5\pi\int_0^1(1+z^2)dz=\frac{20\pi}{3}$

③ $\bar{x}=0$, $\bar{y}=0$

$\bar{z}=\iiint_\Omega z\,dv \Big/ \iiint_\Omega dv$.

$\iiint_\Omega z\,dv=\int_0^1 z\,dz\iint_{D_z}dxdy=5\pi\int_0^1(z+z^3)dz=\frac{15\pi}{4}$

2. 退化 — 平面:

① 点法:

$M_0(x_0,y_0,z_0)\in\pi$, $\vec{n}=\{A,B,C\}\perp\pi$

$\pi: A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

② 一般:

$\pi: Ax+By+Cz+D=0$

③ 截距:

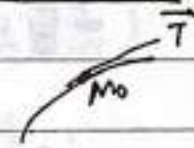
$\pi: \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

3. 空间曲面：

$\Sigma: F(x,y,z)=0.\quad M_0(x_0,y_0,z_0)\in\Sigma$

$\vec{n}=\{F_x', F_y', F_z'\}_{M_0}$

线切面法



(二) 空间曲线.

1. 形式：

① 一般：

$$\Gamma:\begin{cases}F(x,y,z)=0\\ G(x,y,z)=0\end{cases}$$

② 参数：

$$\Gamma:\begin{cases}x=\varphi(t)\\ y=\psi(t)\\ z=\omega(t)\end{cases}$$

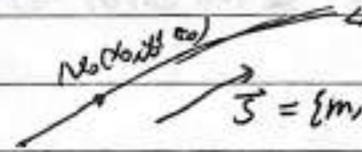
$L: \frac{x-3}{2}=\frac{2y-1}{3}=\frac{z}{1}$

$\vec{s}=\{2, \frac{3}{2}, 1\}$

2. 退化—直线.

① 点向式：

$$L: \frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}$$



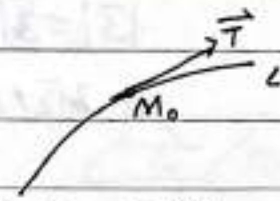
② 参数式：

$$L:\begin{cases}x=x_0+mt\\ y=y_0+nt\\ z=z_0+pt\end{cases}$$

③ 一般式：

$$L:\begin{cases}A_1x+B_1y+C_1z+D_1=0\\ A_2x+B_2y+C_2z+D_2=0\end{cases}$$

3. 空间曲线：



① $L:\begin{cases}x=\varphi(t)\\ y=\psi(t)\\ z=\omega(t)\end{cases}$. $t=t_0$　　$\vec{T}=\{\varphi'(t_0), \psi(t_0), \omega'(t_0)\}$

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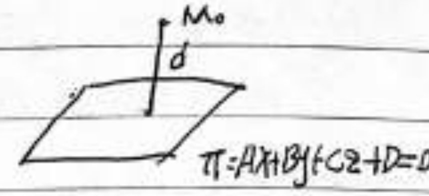
② $L: \begin{cases} F(x,y,z)=0. \\ G(x,y,z)=0 \end{cases}$, $M_0(x_0,y_0,z_0)\in L$.

$\vec{T}=\{F'_x, F'_y, F'_z\}\times\{G'_x, G'_y, G'_z\}$

(三) 距离

1. 点面之距



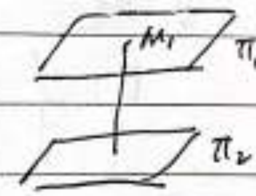
$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

2. 平行面之距.

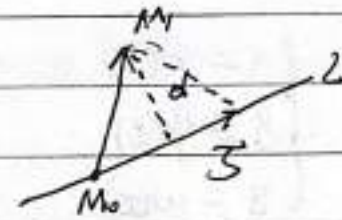
$\pi_1: Ax+By+Cz+D_1=0$

$\pi_2: Ax+By+Cz+D_2=0.$

$$d=\frac{|D_1-D_2|}{\sqrt{A^2+B^2+C^2}}$$



3. 点线之距



$|\overrightarrow{M_0M_1}\times\vec{s}| = \cancel{2}\, S_\triangle$

$\quad\|$

$|\vec{s}|\cdot d = 2S_\triangle$

例2. 求 $M_1(1,0,-1)$ 到直线 L: $\cancel{\frac{x}{2}=\frac{y-2}{1}=\frac{z}{-1}}$ 之距

$L: \begin{cases} x-y=2 \\ x+y+2z=4 \end{cases}$

解: 1° $M_0(1,-1,2)\in L$.

$\vec{s}=\{1,-1,0\}\times\{1,1,2\}=\{-2,-2,2\}$

2° $\overrightarrow{M_0M_1}=\{0,1,-3\}$

$\overrightarrow{M_0M_1}\times\vec{s}=\{-4,6,2\}$

3° $|\overrightarrow{M_0M_1}\times\vec{s}|=\sqrt{16+36+4}=2\sqrt{14}$.

$|\vec{s}|=2\sqrt{3}$

由 $2\sqrt{3}\cdot d=2\sqrt{14} \Rightarrow d=\sqrt{\frac{14}{3}}$

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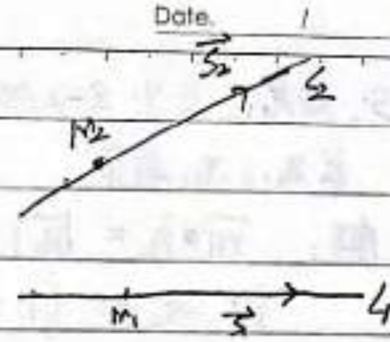
4. 异面直线之距



注：

① L_1, L_2 共面 $\Leftrightarrow \vec{S_1}\times\vec{S_2} \perp \overrightarrow{M_1M_2}$

$\Leftrightarrow (\vec{S_1}\times\vec{S_2})\cdot\overrightarrow{M_1M_2}=0.$

② L_1, L_2 异面 $\Leftrightarrow (\vec{S_1}\times\vec{S_2})\cdot\overrightarrow{M_1M_2}\neq 0$

例3. $L_1: \frac{x}{1}=\frac{y-1}{0}=\frac{z+1}{-1}$, $L_2: \frac{x-1}{2}=\frac{y}{1}=\frac{z-1}{1}$

① L_1, L_2 异面？ ② 若异面求 d？

解：① $M_1(0,1,-1)\in L_1$, $\vec{S_1}=\{1,0,-1\}$.

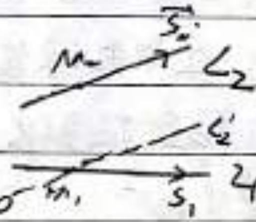
$M_2(1,0,1)\in L_2$, $\vec{S_2}=\{2,1,1\}$.

$\vec{S_1}\times\vec{S_2}=\{1,-3,1\}$. $\overrightarrow{M_1M_2}=\{1,-1,2\}$

$\because (\vec{S_1}\times\vec{S_2})\cdot\overrightarrow{M_1M_2}=1+3+2\neq 0$.

$\therefore L_1, L_2$ 异面

② 过 M_1 作 $L_2'\parallel L_2$



L_1 与 L_2' 所成平面为 π.

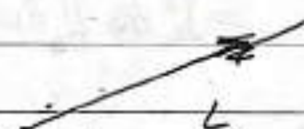
$\pi: 1\times(x-0)-3\times(y-1)+1\times(z+1)=0$

即 $\pi: x-3y+z+4=0$

$$d=\frac{|1+1+4|}{\sqrt{1+9+1}}=\frac{6}{\sqrt{11}}$$

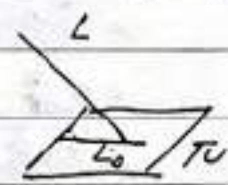
注：平面束.

$$L:\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}$$



$\pi: A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0$

例4. 求 $L:\begin{cases}x-y+z-1=0\\x+y-2z-3=0\end{cases}$ 在 $\pi: x+y+z-4=0$ 上的投影直线.



解：过 L 平面束.

$\pi_0: x-y+z-1+\lambda(x+y-2z-3)=0.$

即 $\pi_0: (\lambda+1)x+(\lambda-1)y+(1-2\lambda)z-1-3\lambda=0$

由 $\{\lambda+1, \lambda-1, 1-2\lambda\}\cdot\{1,1,1\}=0 \Rightarrow \lambda=1$.

$\therefore \pi_0: 2x-2z-4=0$

$\therefore L_0\begin{cases}x-z-2=0\\x+y+z-4=0\end{cases}$

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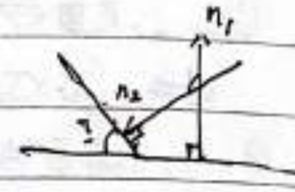
例5. ① $\pi_1: x-y-z-2=0$, $\pi_2: x+y+2z-1=0$

求π_1, π_2夹角.

解: $\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}|\cdot|\vec{n_2}|\cos\theta$



$$\{1,-1,-1\}\cdot\{1,1,2\} = \sqrt{3}\cdot\sqrt{6}\cos\theta.$$

$$\cos\theta = \frac{-2}{3\sqrt{2}} = -\frac{\sqrt{2}}{3}$$

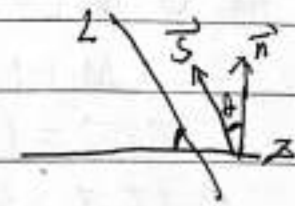
夹角: $\pi - \arccos(-\frac{\sqrt{2}}{3})$

② $L: \frac{x-1}{1} = \frac{y+1}{2} = \frac{z}{0}$. $\pi: x-y-z-2=0$

$\vec{s}=\{1,2,0\}$ $\vec{n}=\{1,-1,-1\}$

$$\cos\theta = \frac{|\vec{s}\cdot\vec{n}|}{|\vec{s}||\vec{n}|}$$

$\frac{\pi}{2}-\theta$

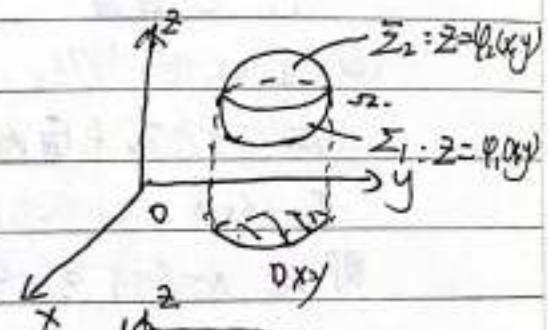


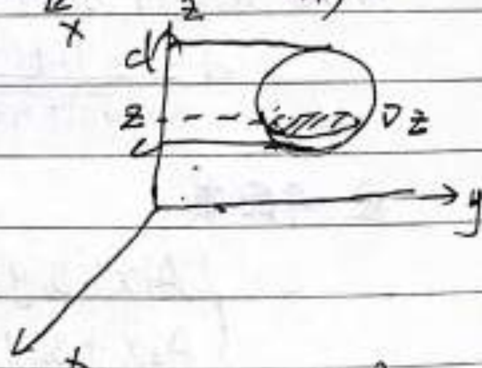
专题二. 三重积分计算法.

法一: 直角坐标法.

① 铅直投影法.

$$\iiint_\Omega f\,dv = \iint_{D_{xy}} dxdy \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f\,dz$$



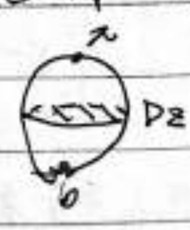


②

$$\iiint_\Omega f\,dv = \int_c^d dz \iint_{D_z} f(x,y,z)\,dxdy$$

例1. 求 $\Omega: (x-z)^2+y^2 = z\sin^2 z$ 介于 $z=0$ 与 $z=\pi$ 之间的体积

解. $V=\iiint_\Omega dv$



$$= \int_0^\pi dz \iint_{D_z} 1\,dxdy$$

$$= \pi\int_0^\pi z\sin^2 z\,dz$$

$$= \pi\times\frac{\pi}{2}\int_0^\pi \sin^2 z\,dz$$

$$= \frac{\pi}{2}\times 2\times\int_0^{\frac{\pi}{2}} \sin^2 z\,dz = \pi^2\times\frac{\pi}{4}$$

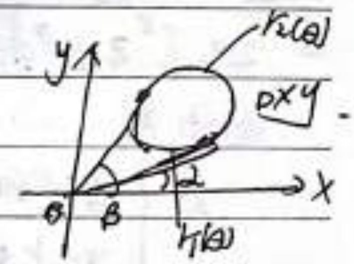
法二：柱面坐标变换法.

1° 特征：① Ω 的边界曲面含 x^2+y^2；

② $f(x,y,z)$ 含 x^2+y^2.

2° 用法：

第一步：投影法 $\begin{cases}(x,y)\in D_{xy}:\\ \varphi_1(x,y)\le z\le \varphi_2(x,y)\end{cases}$

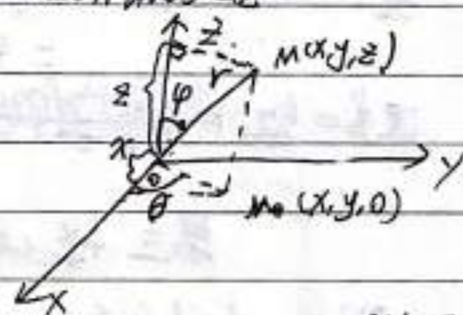


第二步：令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=z\end{cases}$

$\alpha\le\theta\le\beta,\ r_1(\theta)\le r\le r_2(\theta)\quad \varphi_1(r\cos\theta,r\sin\theta)\le z\le\varphi_2(r\cos\theta,r\sin\theta)$

$\iiint_\Omega f\,dv=\int_\alpha^\beta d\theta\int_{r_1(\theta)}^{r_2(\theta)} r\,dr\int_{\varphi_1}^{\varphi_2} f(r\cos\theta,r\sin\theta)\,dz$

法三：球面法.



1° 特征：

① Ω 边界含 $x^2+y^2+z^2$；

② f 含 $x^2+y^2+z^2$.

2° 变换：

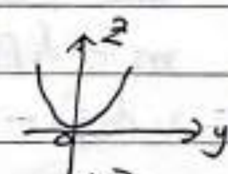
$OM_0=r\sin\varphi$

$\begin{cases}x=r\sin\varphi\cos\theta\\ y=r\sin\varphi\sin\theta\\ z=r\cos\varphi\end{cases}$

3° $dv=r^2\sin\varphi\,dr\,d\theta\,d\varphi$

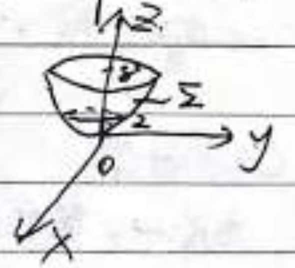
P_{178} 例1. 解.

$L:\begin{cases}y^2=2z\\ x=0\end{cases}\Rightarrow \Sigma:\ 2z=x^2+y^2.$



$I=\iiint_\Omega (x^2+y^2)\,dv$

$=\int_2^8 dz\iint_{x^2+y^2\le 2z} x^2+y^2\,dx\,dz$



$=\int_2^8 dz\int_0^{2\pi} d\theta\int_0^{\sqrt{2z}} r^3\,dr$

$=2\pi\int_2^8 z^2\,dz$

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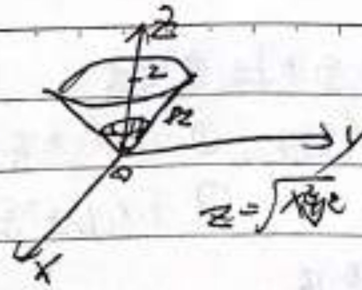
P179 例2. 解:



$\iiint_\Omega \frac{e^z}{\sqrt{x^2+y^2}} dv$

$= \int_0^2 dz \iint_{x^2+y^2 \le z^2} \frac{1}{\sqrt{x^2+y^2}} dxdy$

$= \int_0^2 e^z dz \int_0^{2\pi} d\theta \int_0^z dr$

$= 2\pi \int_0^2 z e^z dz$.

例4. 解.

令 $\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$ $\left(\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \\ 0 \le r \le t \end{array} \right)$

$\iiint_\Omega f(\sqrt{x^2+y^2+z^2}) dv = \int_0^{2\pi} d\theta \int_0^\pi d\varphi \int_0^t r^2 \sin\varphi f(r) dr$

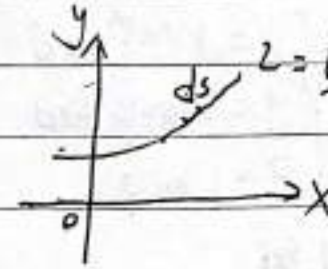
$= 4\pi \int_0^t r^2 f(r) dr$.

原式 $= \lim_{t\to 0} \frac{4\pi\int_0^t r^2 f(r)dr}{\pi t^4} = \lim_{t\to 0} \frac{4\pi t^2 f(t)}{4\pi t^3} = \lim_{t\to 0} \frac{f(t)-f(0)}{t} = f'(0)$

第三模块 (续)

Part IV 4. 曲线积分.

一、对弧长曲线积分.



L: $\rho(x,y)$

(一) 应用背景: 求 m.

1° $\forall ds \subset L$;

2° $dm = \rho(x,y) ds$;

3° $m = \int_L \rho(x,y) ds$.

(二) def — $\int_L f(x,y) ds$

$f(x,y)$ 在 L 上对弧长的曲线积分

(三) 计算

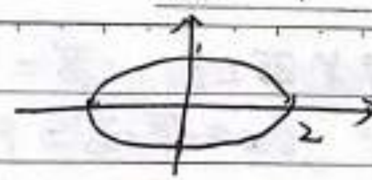
方法一: 替代法.

例1. L: $\frac{x^2}{4} + y^2 = 1$. L 长度为 l 求 $\int_L (x-2y)^2 ds$.

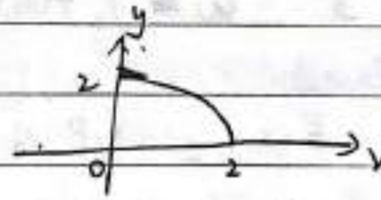
解：$\int_L (x-2y)^2 ds = \int_L (x^2+4y^2-4xy)ds$

$= \int_L (x^2+4y^2)ds = 4\int_L (\frac{x^2}{4}+y^2)ds$

$= 4\int_L 1 ds = 4l$



例2. $I=\int_L (x^2+3y^2)ds$.



解：$I=\int_L (y^2+3x^2)ds$

$2I = 4\int_L (x^2+y^2)ds$

$\Rightarrow I = 2\int_L (x^2+y^2)ds = 8\int_L 1ds = 8\pi$

方法二：定积分法.

Case1. $L: y=\varphi(x) \quad (a\le x\le b)$

$\int_L f(x,y)ds = \int_a^b f[x,\varphi(x)]\cdot\sqrt{1+\varphi'^2(x)}\,dx$

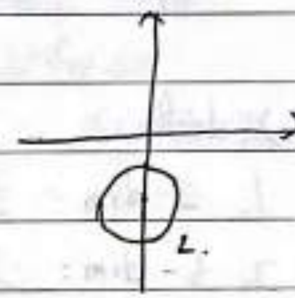
Case2. $L: \begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \quad (\alpha\le t\le\beta)$.

又以参数式.

$\int_L f(x,y)ds = \int_\alpha^\beta f[\varphi(t),\psi(t)]\cdot\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$.

$P_{(232)}$ 例3 解：

$L: x^2+(y+2)^2=1$



1° $I = \int_L (3x-4y)ds = -4\int_L y ds$.

2° $L: \begin{cases} x=\cos\theta \\ y=-2+\sin\theta \end{cases} \quad (0\le\theta\le 2\pi)$

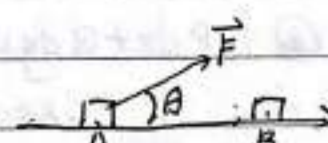
3° $I = -4\int_0^{2\pi}(-2+\sin\theta)\cdot\sqrt{\sin^2\theta+\cos^2\theta}\,d\theta$

$= 16\pi + 4\cos\theta\Big|_0^{2\pi} = 16\pi$

二、对坐标的曲线积分.

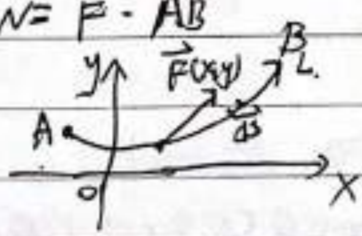
(一) 应用背景：做功.

Case1. $W = \vec{F}\cdot\vec{AB}$



$|\vec{F}|\cos\theta\cdot|\vec{AB}|$.

Case2.



$\vec{F}(x,y) = \{P(x,y), Q(x,y)\} \qquad W=?$

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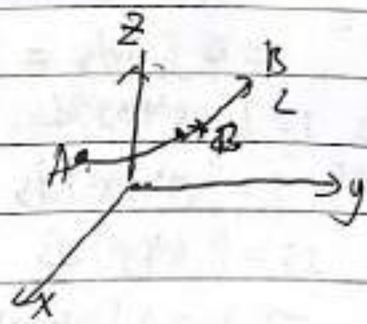
1° $\forall \vec{ds} \subset L, \quad \vec{ds} = \{dx, dy\}$;

2° $dw = \vec{F} \cdot \vec{ds} = P(x,y)dx + Q(x,y)dy$;

3° $w = \int_L P(x,y)\,dx + Q(x,y)\,dy$.



Case 3.

$\vec{F}(x,y,z) = \{P, Q, R\} \quad w = ?$

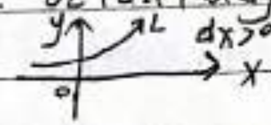
1° $\forall \vec{ds} \subset L$.

$\vec{ds} = \{dx, dy, dz\}$.

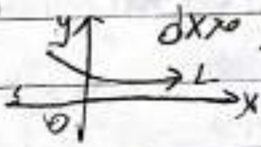
2° $dw = \vec{F} \cdot \vec{ds} = Pdx + Qdy + Rdz$,

3° $w = \int_L Pdx + Qdy + Rdz$

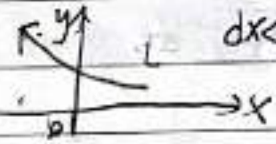
Note: $\int_L Pdx + Qdy$.



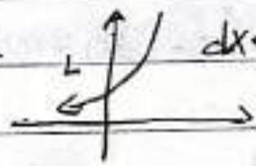
$dx > 0 \quad dy > 0$



$dx > 0 \quad dy < 0$



$dx < 0, \ dy > 0$.

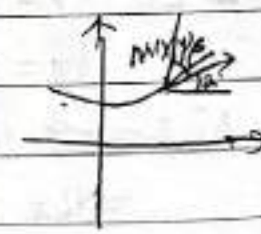


$dx < 0; \ dy < 0$

(二) def s:

1. 2-dim: $\int_L Pdx + Qdy$

2. 3-dim: $\int_L Pdx + Qdy + Rdz$



(三) 性质:

1. $\int_{L^-} = -\int_L$

2. ① $\int_L Pdx + Qdy = \int_L (P\cos\alpha + Q\cos\beta)ds$.

② $\int_L Pdx + Qdy + Rdz = \int_L (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,ds$

(四) 2-dim: $\int_L Pdx + Qdy$ 计算法.

方法一: 定积分法.

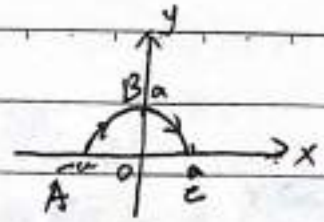
Case 1. $L: y = \varphi(x)$ (起点 $x=a$, 终点 $x=b$)

$$\int_L Pdx + Qdy = \int_a^b P[x, \varphi(x)]\,dx + Q[x, \varphi(x)] \cdot \varphi'(x)\,dx$$

Case 2. $L: \begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$ (起点 $t=\alpha$, 终 $t=\beta$)

$$\int_L Pdx + Qdy = \int_\alpha^\beta P[\varphi(t), \psi(t)]\,\varphi'(t)\,dt + Q[\varphi(t), \psi(t)]\,\psi'(t)\,dt$$

P232 例1. 解:

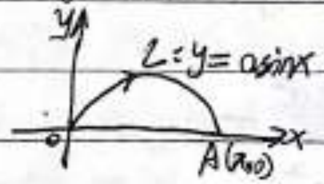


令 $L:\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$ （起 $t=\pi$, 终 $t=0$）

$$I = \int_L (x^2+y^2)dx - x\,dy = \int_L a^2\,dx - x\,dy = \int_\pi^0 a^2\times(-a\sin t)dt - a\cos t\,a\cos t\,dt$$

$$= \int_0^\pi (a^3\sin t + a^2\cos^2 t)\,dt = 2a^3 + 2a^2\times\frac{\pi}{4}$$

例2. 解:



$L: y = a\sin x$

（起 $x=0$, 终 $x=\pi$）

$$I(a) = \int_L (1+y^3)dx + (2x+y)dy = \int_0^\pi (1+a^3\sin^3 x)dx + (2x+a\sin x)a\cos x\,dx$$

$$= \pi + \frac{4}{3}a^3 + 2a\int_0^\pi x\,d(\sin x) + a^2\int_0^\pi \sin x\,d(\sin x)$$

$$= \pi + \frac{4}{3}a^3 + 2a\left(x\sin x\Big|_0^\pi - \int_0^\pi \sin x\,dx\right)$$

$$= \pi + \frac{4}{3}a^3 - 4a.$$

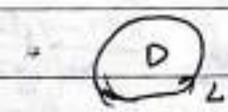
令 $I'(a) = 4a^2 - 4 = 0 \Rightarrow a = 1$ $\qquad I''(a) = 8a$

$\because I''(1) = 8 > 0$ $\therefore$ $a=1$ 时 $I(a)$ 最小

方法二：二重积分法

$F(b) - F(a) = \int_a^b f(x)\,dx$

a —— b 边 　a、b是一界



Th (Green)

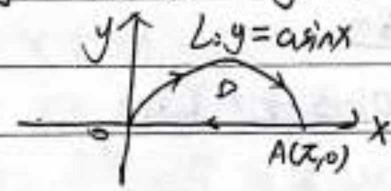
If ① D 一连通, L为D的正向边界.

② $P(x,y)$、$Q(x,y)$ 在D上连续

可偏导

则 $\oint_L P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)d\sigma$

P232 例2 解:



$\overline{OA}: y = 0$

（起 $x=0$, 终 $x=\pi$）

1° $P = 1+y^3$ $\quad Q = 2x+y$ $\qquad \frac{\partial P}{\partial y} = 3y^2$ $\qquad \frac{\partial Q}{\partial x} = 2.$

2° $I = \int_L = \int_{L+\overline{AO}} + \int_{\overline{OA}}$

$$\int_{L+\overline{AO}} = -\iint_D (2-3y^2)\,d\sigma = \iint_D (3y^2-2)\,d\sigma = \int_0^\pi dx\int_0^{a\sin x}(3y^2-2)\,dy$$

$$= \int_0^\pi (a^3\sin^3 x - 2a\sin x)\,dx = \frac{4}{3}a^3 - 4a$$

$\int_{\overline{OA}} = \int_0^\pi 1\,dx = \pi$ $\qquad I = \frac{4}{3}a^3 - 4a + \pi$

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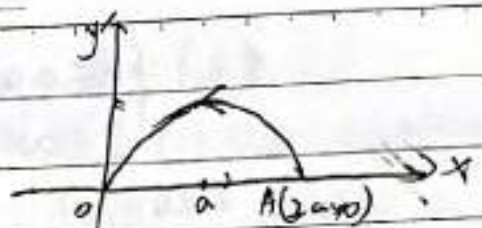
P_{233} 例3. 解：



1° $P=e^x\sin y-b(x+y)$ $\quad Q=e^x\cos y-ax$

$\frac{\partial P}{\partial y}=e^x\cos y-b$ $\quad \frac{\partial Q}{\partial x}=e^x\cos y-a$

2° $I=\int_L=\int_{L+\overline{OA}}-\int_{\overline{OA}}$

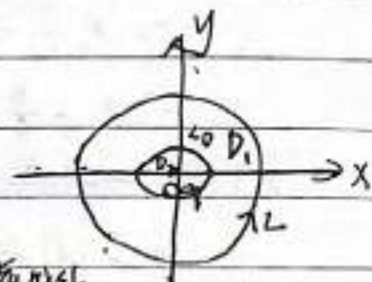
$\int_{L+\overline{OA}}=\iint(b-a)\,d\sigma=\frac{\pi a^2}{2}(b-a)$

$\int_{\overline{OA}}=\int_0^{2a}(-bx)\,dx=-b\times 2a^2$

$\therefore I=\frac{\pi}{2}a^2(b-a)+2a^2b$

补. 例4. $I=\oint_L\frac{xdy-ydx}{x^2+4y^2}$，L为绕O的正向闭曲线.

解：



1° $P=-\frac{y}{x^2+4y^2}$，$Q=\frac{x}{x^2+4y^2}$

$\frac{\partial P}{\partial y}=\frac{4y^2-x^2}{(x^2+4y^2)^2}$ $\quad \frac{\partial Q}{\partial x}=\frac{4y^2-x^2}{(x^2+4y^2)^2}$

$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ $\left((x,y)\neq(0,0)\right)$

2° $L_0: x^2+4y^2=r^2$ ($r>0$, L包L_0在内. L_0逆时针) 如图.

3° $\oint_{L+L_0^-}=\iint_{D_1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma=0$

$\Rightarrow \oint_L-\oint_{L_0}=0$

$I=\oint_L=\oint_{L_0}\frac{xdy-ydx}{x^2+4y^2}=\frac{1}{r^2}\oint_{L_0}xdy-ydx$

$=\frac{2}{r^2}\iint_{D_2}1\,d\sigma=\frac{2}{r^2}\times\pi\times r\times\frac{r}{2}=\pi$

方法三：与路径无关问题

Th. D一单连通，P、Q在D上连续偏导数.

以下命题相互等价：

① $\int_L Pdx+Qdy$与路径无关；

② $\forall$闭曲线$C\subset D$ 有 $\oint_C Pdx+Qdy=0$

③ $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ $\quad\left(\frac{\partial u}{\partial x}=P,\ \frac{\partial u}{\partial y}=Q\right)$

④ $\exists u(x,y)$ 使 $du=Pdx+Qdy$

Notes:

① If $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$, 则

$$\int_L Pdx+Qdy=\int_{(x_0,y_0)}^{(x_1,y_1)}Pdx+Qdy$$

$$=\int_{x_0}^{x_1}P(x,y_0)\,dx+\int_{y_0}^{y_1}Q(x_1,y)\,dy$$

② If $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$，且 $Pdx+Qdy=du$

则 $\int_L Pdx+Qdy=u(x_1,y_1)-u(x_0,y_0)$.

③ If $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$，则

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)}Pdx+Qdy=\int_{x_0}^{x}P(x,y_0)\,dx+\int_{y_0}^{y}Q(x,y)\,dy$$

④ 全微分方程

$$P(x,y)\,dx+Q(x,y)\,dy=0$$

If $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$，称 —— 为全 ——

例1. 求 $(xy^2+1)dx+(x^2y+2y)dy=0$ 通解.

解：$P=xy^2+1\quad Q=x^2y+2y$

$$\frac{\partial Q}{\partial x}=2xy=\frac{\partial P}{\partial y}$$

法一：$(xy^2dx+x^2ydy)+dx+2ydy=0$.

$\Rightarrow d(\frac{1}{2}x^2y^2+x+y^2)=0$

通解：$\frac{1}{2}x^2y^2+x+y^2=C$

$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$

$Pdx+Qdy=du$

$Pdx+Qdy=0$

$\Updownarrow$

$du=0$

$\Updownarrow$

$u=C$

法二：$u(x,y)=\int_{(0,0)}^{(x,y)}(xy^2+1)dx+(x^2y+2y)dy$

$$=\int_0^x 1\,dx+\int_0^y(x^2y+2y)\,dy$$

$$=x+\frac{1}{2}x^2y^2+y^2$$

$\therefore$ 通解 $x+\frac{1}{2}x^2y^2+y^2=C$

例2. 验证 $x>0$ 内，$\frac{xdy-ydx}{4x^2+y^2}$ 为某 $u(x,y)$ 的全微分，求 $u(x,y)$.

证：$P=-\frac{y}{4x^2+y^2}\quad Q=\frac{x}{4x^2+y^2}$

$\because \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=?$

$\therefore \exists\, u(x,y)$ 使 $du=\frac{xdy-ydx}{4x^2+y^2}$

$$u(x,y)=\int_{(1,0)}^{(x,y)}\frac{xdy-ydx}{4x^2+y^2}=\int_1^x 0\,dx+\int_0^y\frac{x\,dy}{(2x)^2+y^2}=x\times\frac{1}{2x}\arctan\frac{y}{2x}\Big|_0^y$$

$$=\frac{1}{2}\arctan\frac{y}{2x}$$

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例3. $\varphi(x)$ 可导. $\varphi(0)=2$.

$\int_L xy^2dx+\varphi(x)y\,dy$ 与路径无关

① 求 $\varphi(x)$; ② $\int_{(2,2)}^{(3,4)} xy^2dx+\varphi(x)y\,dy=?$

解: ① $P=xy^2$ $Q=\varphi(x)y$

$\Rightarrow \varphi'(x)y=2xy \Rightarrow \varphi'(x)=2x$

$\Rightarrow \varphi(x)=x^2+C$

$\because \varphi(0)=2$ $\therefore \varphi(x)=x^2+2$

② $I=\int_{(2,2)}^{(3,4)} xy^2dx+(x^2+2)y\,dy$

$=\int_{2,2}^{3,4} d(\frac{1}{2}x^2y^2+y^2)$

$=(\frac{1}{2}x^2y^2+y^2)\Big|_{(2,2)}^{(3,4)}$

P233. 例4. 解:

1° $\frac{\partial Q}{\partial x}=2x \Rightarrow Q=x^2+\varphi(y)$

2° 左 $=\int_{(0,0)}^{(t,1)} 2xy\,dx+[x^2+\varphi(y)]dy$

$=\int_0^t 0\,dx+\int_0^1[t^2+\varphi(y)]dy=t^2+\int_0^1\varphi(y)dy$

右 $=\int_{(0,0)}^{(1,t)} 2xy\,dx+[x^2+\varphi(y)]dy$

$=\int_0^1 0\,dx+\int_0^t[1+\varphi(y)]dy=t+\int_0^t\varphi(y)dy.$

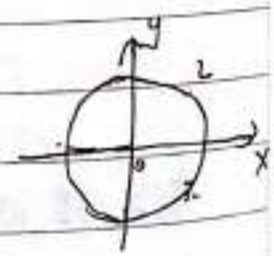
3° $t^2+\int_0^1\varphi(y)dy=t+\int_0^t\varphi(y)dy$

$\Rightarrow 2t=1+\varphi(t) \Rightarrow \varphi(t)=2t-1$

$\varphi(y)=2y-1$

$Q=x^2+2y-1$

例5. $\varphi(x)$ 可导. $\varphi(1)=1$ $\oint_L \frac{x\,dy-y\,dx}{\varphi(x)+y^2}=A$ $(A\neq 0)$

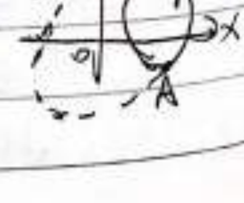


① 求 $\varphi(x)$; ② 求 A

解: ① $P=-\frac{y}{\varphi(x)+y^2}$ $Q=\frac{x}{\varphi(x)+y^2}$

$\frac{\partial Q}{\partial x}=\frac{\varphi(x)+y^2-x\varphi'(x)}{(\ \ )^2}$ $\frac{\partial P}{\partial y}=\frac{y^2-\varphi(x)}{(\ \ )^2}$

任取 L 不绕 O 正向



$\oint_{L_2+L_1^-}=A$ 由此 $\oint_L=0$

$\oint_{L_3+L_2}=A$

$\varphi(x)-x\varphi'(x)=-\varphi(x)$

$\Rightarrow x\varphi'(x)-2\varphi(x)=0 \Rightarrow \varphi'(x)=\frac{2}{x}\varphi(x)^{20}$

$$\varphi(x)=Ce^{-\int\frac{2}{x}dx}=Cx^2$$

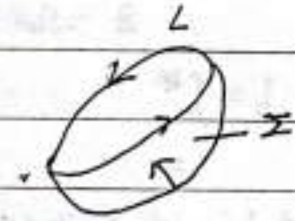
$\because \varphi(1)=1$ $\quad\therefore \varphi(x)=x^2$

② $A=\oint_L \frac{x\,dy-y\,dx}{x^2+y^2}$

L_0:

(五) 3-dim：$\int_L P\,dx+Q\,dy+R\,dz$.

方法一：stokes 法



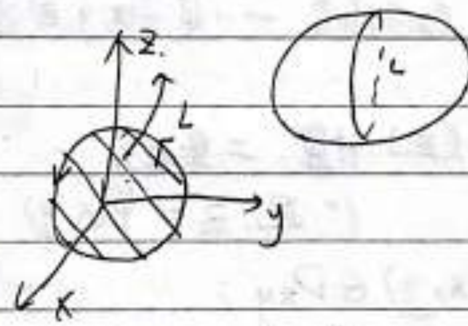
右手

$$\oint_L P\,dx+Q\,dy+R\,dz$$

$$=\iint_\Sigma \begin{vmatrix} dy\,dz & dz\,dx & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$=\iint_\Sigma \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} ds$$

P_{236}. 例1. 解：



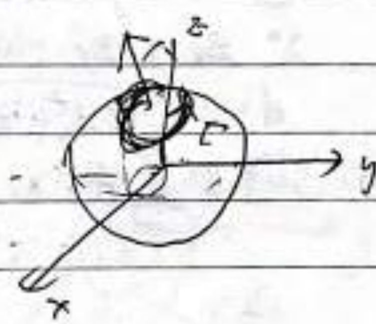
1°. Σ：$x+y+z=0$，上；

2°. $\vec{n}=\{1,1,1\}$ $\quad\cos\alpha=\cos\beta=\cos\gamma=\frac{1}{\sqrt3}$

3° $I=\frac{1}{\sqrt3}\iint_\Sigma \begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} ds$

$=-\sqrt3\iint_\Sigma ds\;=-\sqrt3\pi a^2$

例2. 解：



1° Σ：$x^2+y^2+z^2=1$ 上侧

2° $\vec{n}=\{2x,2y,2z\}$

$\cos\alpha=x\quad\cos\beta=y\quad\cos\gamma=z$

3° $I=\iint_\Sigma \begin{vmatrix} x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} ds=-2\iint_\Sigma(xy+xz+yz)\,ds$

$=-2\iint_\Sigma xz\,ds$

Date. /

法二：定积分法

例 $I=\int_L y\,dx+2xz\,dy-4\,dz$

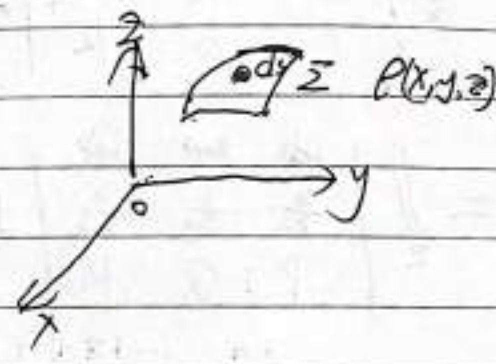
$L:\begin{cases}x^2+y^2=1\\ x+y-z=2\end{cases}$ 从z轴正向看逆.

解：令 $\begin{cases}x=\cos t\\ y=\sin t\\ z=\sin t+\cos t-2\end{cases}$ （起：$t=0$，终 $t=2\pi$）

$I=\int_0^{2\pi}$

Part V 曲面积分

一、对面积曲面积分



(一) 背景：m.

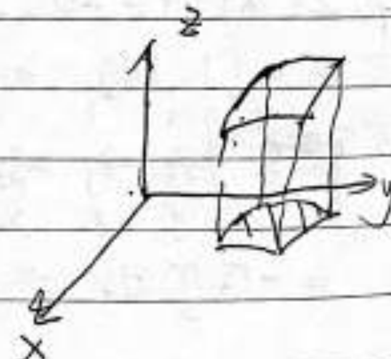
1° $\forall\, ds\subset\Sigma$；

2° $dm=\rho(x,y,z)\,ds$.

3° $m=\iint_\Sigma \rho(x,y,z)\,ds$.

(二) def — $\iint_\Sigma f(x,y,z)\,ds$.

(三) 计算：二重积分



1° $\Sigma: z=\varphi(x,y)$

$(x,y)\in D_{xy}$；

2° z'_x，z'_y

$ds=\sqrt{1+z_x'^2+z_y'^2}\,d\sigma$

3° $I=\iint_{D_{xy}} f[x,y,\varphi(x,y)]$

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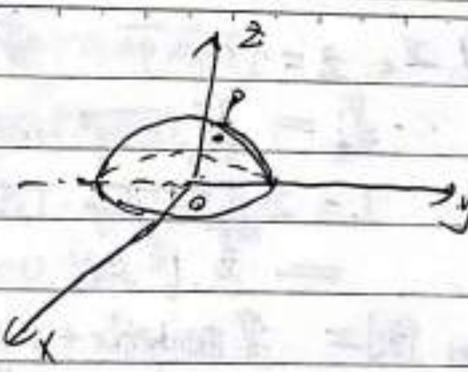
P_{239}. 例4. 解: $P(x,y,z)\in\Sigma$



$\vec{n}=\{x,y,2z\}$

π: $x(X-x)+y(Y-y)+2z(Z-z)=0$

即π: $\frac{x}{2}X+\frac{y}{2}Y+zZ-1=0$

$d=\frac{1}{\sqrt{\frac{x^2}{4}+\frac{y^2}{4}+z^2}}$

$I=\iint_\Sigma \frac{z}{d}ds=\iint_\Sigma z\sqrt{\frac{x^2}{4}+\frac{y^2}{4}+z^2}\,ds$

1°. Σ: $z=\sqrt{1-\frac{x^2}{2}-\frac{y^2}{2}}$ $\quad D_{xy}=x^2+y^2\le 2$.

2°. $z'_x=\frac{-x}{2\sqrt{\ }}$ $\quad z'_y=\frac{-y}{2\sqrt{\ }}$

$ds=\sqrt{1+\frac{x^2+y^2}{4(1-\frac{x^2}{2}-\frac{y^2}{2})}}\,d\sigma=\frac{\sqrt{4-x^2-y^2}}{2z}d\sigma$

3° $I=\iint_{D_{xy}}\sqrt{1-\frac{x^2}{4}-\frac{y^2}{4}}\cdot\frac{1}{2}\sqrt{4-x^2-y^2}\,d\sigma$

$=\frac{1}{4}\iint_{D_{xy}}(4-x^2-y^2)d\sigma \quad =\frac{1}{4}\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}r(4-r^2)dr$

二. 对坐标的曲面积分.

$\iint_\Sigma P\,dydz+Q\,dzdx+R\,dxdy$.

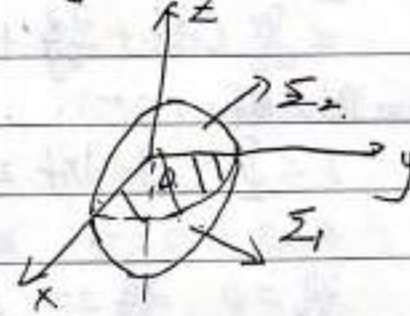
方法一:

$\iint_\Sigma R\,dxdy$:

1° Σ: $z=\varphi(x,y)$. $(x,y)\in D_{xy}$;

2° $\iint_\Sigma R\,dxdy=\pm\iint_{D_{xy}}R[x,y,\varphi(x,y)]\,dxdy$

例1. $\iint_\Sigma z\cdot\sqrt{x^2+y^2}\,dxdy$



解: $\iint_\Sigma=\iint_{\Sigma_1}+\iint_{\Sigma_2}$

① Σ_1: $z=-\sqrt{1-x^2-y^2}$

D_{xy}: $x^2+y^2\le 1$ $(x\ge 0, y\ge 0)$

$\iint_{\Sigma_1}=-\iint_{D_{xy}}-\sqrt{1-x^2-y^2}\cdot\sqrt{x^2+y^2}\,d\sigma=\iint_{D_{xy}}\sqrt{1-x^2-y^2}\cdot\sqrt{x^2+y^2}\,d\sigma$.

奇函数两倍，偶函数为0.

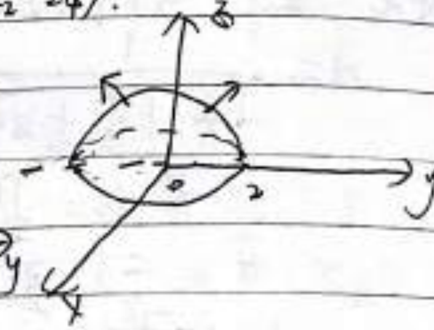
Date. /

② $\Sigma_2: z=\sqrt{1-x^2-y^2}$ 　 $D_{xy}: x^2+y^2\le 1\ (x\ge 0, y\ge 0)$.

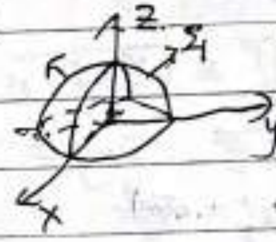
$\iint_{\Sigma_2} = \iint_{D_{xy}} \sqrt{1-x^2-y^2}\cdot\sqrt{x^2+y^2}\, d\sigma$

$I = 2\iint_{D_{xy}} \sqrt{1-x^2-y^2}\cdot\sqrt{x^2+y^2}\, d\sigma = \pi\int_0^1 r^2\sqrt{1-r^2}\,dr$

$= \pi\int_0^{\frac{\pi}{2}} \sin^2 t\,(1-\sin^2 t)\,dt = \pi(I_2-I_4)$.



P241. 例2. $\iint_\Sigma \underbrace{yz\,dz\,dx}_{I_1} + \underbrace{2\,dx\,dy}_{I_2}$



解：① $I_1 = \iint_\Sigma yz\,dx\,dz$.

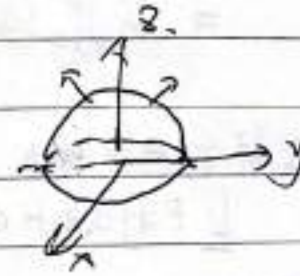
$= 2\iint_{\Sigma_1} yz\,dz\,dx$

$\Sigma_1: y=\sqrt{4-x^2-z^2}$

$D_{xz}: x^2+z^2\le 4\ (z\ge 0)$

$I_1 = 2\iint_{D_{xz}} z\sqrt{4-x^2-z^2}\,dz\,dx$

$= 2\int_0^{\pi} d\theta\int_0^{2} r^2\sin\theta\sqrt{4-r^2}\,dr$



② $I_2 = \iint_\Sigma 2\,dx\,dy$

$\Sigma: z=\sqrt{4-x^2-y^2}$

$D_{xy}: x^2+y^2\le 4$

$I_2 = 2\iint_{D_{xy}} dx\,dy = 2\times 4\pi = 8\pi$

方法二. Gauss.

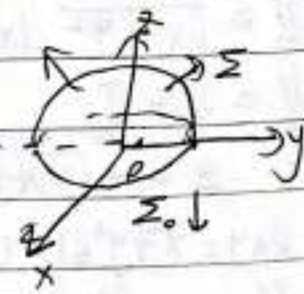
$\oiint_\Sigma P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$

$= \iiint_\Omega \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right) dv$

P, Q, R

P241 例2.解:

$I = \iint_\Sigma yz\,dz\,dx + 2\,dx\,dy$



1° $P=0$. 　 $Q=yz$ 　 $R=2$.

$\frac{\partial P}{\partial x}=0$, $\frac{\partial Q}{\partial y}=z$, $\frac{\partial R}{\partial z}=0$

2° $\Sigma_0: z=0\ (x^2+y^2\le 4)$ 下侧.

$I = \iint_\Sigma = \oiint_{\Sigma+\Sigma_0} - \iint_{\Sigma_0}$

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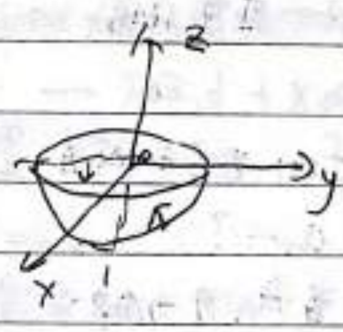
$\oiint_{\Sigma+\Sigma_0} = \iiint_\Omega z\,dv = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^2 r^2\sin\varphi\, r\cos\varphi\, dr$

$= 2\pi \int_0^{\frac{\pi}{2}} \sin\varphi\, d(\sin\varphi) \int_0^2 r^3\,dr$

$= 2\pi \times \frac{1}{2} \times 4 = 4\pi$,

$\iint_{\Sigma_0} = \iint_{\Sigma_0} 2\,dx\,dy = -2\iint_{D_{xy}} 1\,d\sigma = -8\pi$

P242 例4. 解:



$I = \frac{1}{a}\iint_\Sigma \cdots \cdots \frac{dI}{64}$

$P = ax \quad Q = -2y(z+a) \quad R = (z+a)^2$

$\frac{\partial P}{\partial x} = a \quad \frac{\partial Q}{\partial y} = -2(z+a) \quad \frac{\partial R}{\partial z} = 2(z+a)$

~~Σ_1~~ Σ_0 $z=0$ $(x^2+y^2 \le a^2)$ 下.

$I_1 = \oiint_{\Sigma+\Sigma_0} - \iint_{\Sigma_0}$

$\oiint_{\Sigma+\Sigma_0} = -a\iiint_\Omega dv = -\frac{2}{3}\pi a^4$

$\iint_{\Sigma_0} = \iint_{\Sigma_0} a^2\,dx\,dy = -a^2\iint_{D_{xy}} dx\,dy = -a^2 \cdot \pi a^2 = -\pi a^4$

$I_1 = \frac{1}{3}\pi a^4 \qquad I = \frac{1}{3}\pi a^3$

数一强化考点 — Fourier 级数.

Date. /

背景:

单一周期信号: $a\cos n\omega t + b\sin n\omega t$

假设 $f(x)$ 是以 2π 为周期的信号 (并非单一) $f(x)$ 分解:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

$\frac{a_0}{2}$ — 直流成份

$a_1\cos x+b_1\sin x$ —— 一次谐波

Q1. $f(x)$ 可否分解为 $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$?

$a_0=?$ $a_n=?$ $b_n=?$

Q2. 若 $f(x)$ 可分解为 $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$f(x)$ 与 $\frac{a_0}{2}+\sum_{n=1}^{\infty}(\ ---\ )$ 什么关系?

一. $f(x)$ 以 2π 为周期.

Th. 设 $f(x)$ 以 2π 为周期, 若:

① $f(x)$ 在 $[-\pi,\pi)$ 上连续或 $\exists$ 有限个第一类间断点

② $f(x)$ 在 $[-\pi,\pi)$ 上 $\exists$ 有限个极值点. 则

直 1°. $f(x)$ 可以展成 $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$.

其中. $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$.

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$ $(n=1,2,\cdots)$;

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$

2° 若 x 为 $f(x)$ 连续点, 则

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=f(x);$$

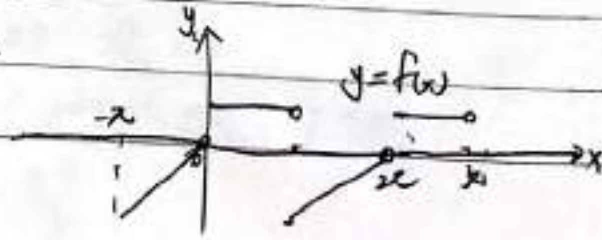
若 x 为 $f(x)$ 间断点, 则

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{f(x-0)+f(x+0)}{2}$$

例1. 设 $f(x)$ 以 2π 为周期, 在 $[-\pi,\pi)$ 上, $f(x)$ 表达式为 ~~f(x)=~~:

$$f(x)=\begin{cases}x, & -\pi\le x<0\\ 1, & 0\le x<\pi\end{cases}$$ 求 $f(x)$ 的 Fourier 级数.

解: 1° 作用图.



$x=k\pi\ (k\in Z)$ 为 $f(x)$ 的间断点;

2° $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}x\,dx+\int_{0}^{\pi}1\,dx\right]=1-\frac{\pi}{2}$;

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}x\cos nx\,dx+\int_{0}^{\pi}\cos nx\,dx\right]$

$=\frac{1-(-1)^n}{n^2\pi}$;

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}x\sin nx\,dx+\int_{0}^{\pi}\sin nx\,dx\right]$

$=\frac{(-1)^{n+1}}{n}+\frac{1-(-1)^n}{n\pi}$

3°. $f(x)=\left(\frac{1}{2}-\frac{\pi}{4}\right)+\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2\pi}\cos nx+\frac{(-1)^{n+1}\pi+1-(-1)^n}{n\pi}\sin nx\right]$

$(-\infty<x<+\infty,\ x\neq k\pi,\ k\in Z)$.

~~当 $x=k\pi\ (k\in Z)$ 时~~ ① 当 $x=2k\pi\ (k\in Z)$ 时

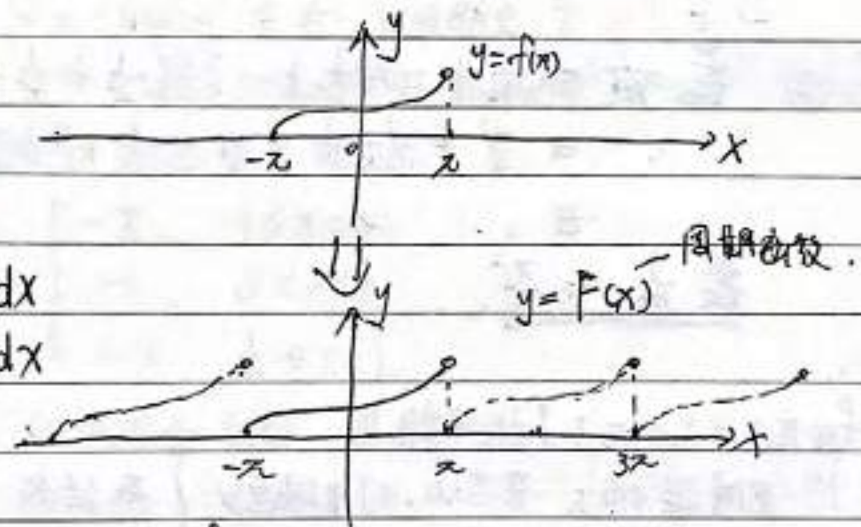
$\left(\frac{1}{2}-\frac{\pi}{4}\right)+\sum_{n=1}^{\infty}(\cdots)=\frac{1}{2}$;

② 当 $x=(2k+1)\pi\ (k\in Z)$ 时,

$\left(\frac{1}{2}-\frac{\pi}{4}\right)+\sum_{n=1}^{\infty}(\text{——})=\frac{1-\pi}{2}$

二. $f(x)$ 定义于 $[-\pi,\pi)$. (非周期)

1°. 周期延拓:



2° $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$

(If $x\in[-\pi,\pi)$ 时, $F(x)=f(x)$).

3° $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$\left(\begin{array}{ll}-\pi<x<\pi, & -\pi\le x<\pi \\ -\pi<x\le\pi, & -\pi\le x\le\pi\end{array}\right)$

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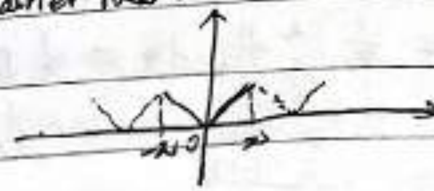
例2. $f(x)=|x|$ $(-\pi\le x\le\pi)$ 展成Fourier级数

解: 1°. 周期延拓:



2°. $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}x\,dx=\pi$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx=\frac{2[(-1)^n-1]}{n^2\pi}=\begin{cases}\frac{-4}{n^2\pi}, & n=1,3,5\cdots\\ 0, & n=2,4,6\cdots\end{cases}$

$b_n=0$ $(n=1,2,\cdots)$.

3° $|x|=\frac{\pi}{2}-\frac{1}{\pi}\left(\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\cdots\right)$

$(-\pi\le x\le\pi)$

Note:

① $x=0$ 时. $0=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$

$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$;

② $\sum_{n=1}^{\infty}\frac{1}{n^2}=\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)+\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right)$

$=\frac{\pi^2}{8}+\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$

故

$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

三、$f(x)$ 定义于 $[0,\pi]$ (非周期)

1° 区间延拓; 在 $[-\pi,0]$ 补充定义 $\begin{cases}\text{奇延拓,}\\ \text{偶延拓,}\end{cases}$ $\Rightarrow$ 周期延拓

2° Case 1. 奇延拓.(展成正弦级数)

$a_0=0$;

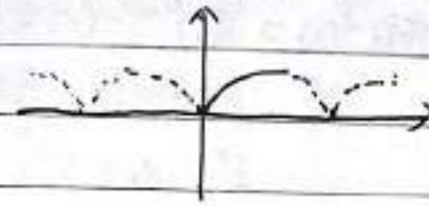
$a_n=0$;

$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$

$f(x)=\underbrace{\sum_{n=1}^{\infty}b_n\sin nx}_{\text{正弦级数}}$ $\left(\begin{array}{ll}0<x<\pi & ;\ 0\le x<\pi\ \checkmark\\ 0<0\le\pi & ,\ 0\le x\le\pi\end{array}\right)$

Date.　　/

Case 2. 偶延拓（展成余弦级数）



$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \underbrace{\sum_{n=1}^{\infty} a_n \cos nx}_{\text{余弦级数}} \quad (0 \le x \le \pi)$$

四. $f(x)$ 以 $2l$ 为周期.　　$\frac{\pi}{l}$

1° 作 $y=f(x)$ 图.

2° $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$;

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$;

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$:

3° $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} \cancel{dx} + b_n \sin\frac{n\pi x}{l}\right)$

$(-\infty < x < +\infty,\ x \ne$? 间断点 $)$

4° $x=$ 间断点时. $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) = \frac{f(x-0)+f(x+0)}{2}$

例. $f(x)$ 以 2 为周期. $x \in [-1, 1)$ 时

$$f(x) = \begin{cases} -x, & -1 \le x < 0. \\ -1, & 0 \le x < \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x < 1. \end{cases}$$

$S(x)$ 为 Fourier 级数的和函数，求 $S(-\frac{5}{2})$

解. $S(-\frac{5}{2}) = S(-2+\frac{1}{2}) = S(\frac{1}{2}) = \frac{f(\frac{1}{2}-0)+f(\frac{1}{2}+0)}{2} = \frac{-1+\frac{1}{2}}{2} = -\frac{1}{4}$

五. $f(x)$ 定义于 $[-l, l)$.

1°. $y=f(x)$ 周期延拓.

2°. $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

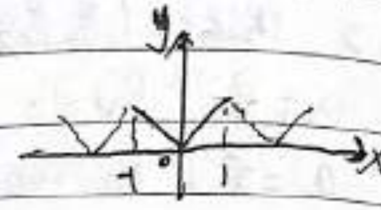
$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

3° $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$ $\left(\begin{array}{l} -l \le x \le l,\ -l \le x < l \\ -l < x < l \ \text{[illegible]} \end{array}\right)$

Date. /

例 $f(x)=|x| \quad (-1\le x<1)$

解：1°. $y=|x|$ 周期延拓.



2°. $a_0=\frac{1}{1}\int_{-1}^{1}f(x)\,dx=2\int_0^1 x\,dx=1$

$a_n=\frac{1}{1}\int_{-1}^{1}|x|\cos\frac{n\pi}{l}x\,dx=2\int_0^1 x\cos n\pi x\,dx$

$=\frac{2}{n\pi}\int_0^1 x\,d(\sin n\pi x)=-\frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx$

$=\frac{2}{n^2\pi^2}\cos n\pi x\Big|_0^1=\frac{2[(-1)^n-1]}{n^2\pi^2}$

$=\begin{cases}-\frac{4}{n^2\pi^2}, & n=1,3,\cdots\\ 0, & n=2,4,\cdots\end{cases}$

$b_n=0$

3° $|x|=\frac{1}{2}-\frac{4}{\pi^2}\left(\frac{1}{1^2}\cos\pi x+\frac{1}{3^2}\cos 3\pi x+\cdots\right)$

$(-1\le x\le 1)$

Notes:

① $x=0$时，$0=\frac{1}{2}-\frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$

$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$

② $S=\sum_{n=1}^{\infty}\frac{1}{n^2}=\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right)+\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right)$

$=\frac{\pi^2}{8}+\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$

$=\frac{\pi^2}{8}+\frac{1}{4}S \Rightarrow S=\frac{\pi^2}{6}$

即 $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

六. f(x) 定义于 [0, l)

(一) 展成余弦级数

1° $y=f(x)$ 偶延拓、周期延拓.

2° $a_0=\frac{2}{l}\int_0^l f(x)\,dx \qquad a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$, $b_n=0$

3° $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l} \qquad (0\le x\le \pi)$.

(二) 展成正弦级数

1° $y=f(x)$ 奇延拓，周期延拓.

2° $a_0=0 \qquad a_n=0 \qquad b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

3° $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ $\left(\begin{matrix} 0\le x\le l & 0\le x<l \\ 0<x\le l & 0<x<l \end{matrix} ?\right)$

数学一专题——贝努利方程、Euler方程

一、贝努利方程

$$\begin{cases} \frac{dy}{dx}+P(x)y=0 \Rightarrow y=Ce^{-\int P(x)dx} \\ \frac{dy}{dx}+P(x)y=Q(x) \Rightarrow y=[\int Q(x)e^{\int P(x)dx}dx+C]e^{-\int P(x)dx} \end{cases}$$

def — $\frac{dy}{dx}+P(x)y=Q(x)y^n$ $(n\neq 0,1)$

解法：

令 $y^{1-n}\triangleq u$，则

$$\frac{du}{dx}+(1-n)P(x)u=(1-n)Q(x)$$

例1. 求 $\frac{dy}{dx}+\frac{2}{x}y=y^2$ 通解.

解：令 $y^{1-2}=u$，则

$$\frac{du}{dx}-\frac{2}{x}u=-1$$

$$u=[\int(-1)e^{\int-\frac{2}{x}dx}dx+C]e^{-\int-\frac{2}{x}dx}$$

$$=(\frac{1}{x}+C)x^2=Cx^2+x$$

$\therefore$ 通解 $y=\frac{1}{Cx^2+x}$

二、Euler方程

(一) def — 形如

$$x^ny^{(n)}+a_1x^{n-1}y^{(n-1)}+\cdots+a_{n-1}xy'+a_ny=f(x)$$

称为欧拉方程

(二) 解法

令 $x=e^t$，$t=\ln x$. $\frac{d}{dt}\triangleq D$

$xy'=Dy=\frac{dy}{dt}$

$x^2y''=D(D-1)y=(D^2-D)y=\frac{d^2y}{dt^2}-\frac{dy}{dt}$.

$x^3y'''=D(D-1)(D-2)y=\cdots$

Date. /

例2. 求 $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \cancel{xe^x}\ 3x^2+1$

解：令 $x=e^t$，$t=\ln x$.

$$x\frac{dy}{dx} = Dy = \frac{dy}{dt}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y = \frac{d^2y}{dt^2} - \frac{dy}{dt} \quad 代入.$$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 3e^{2t}+1$$

1° $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2.$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0 \Rightarrow y = C_1e^t + C_2e^{2t};$$

2° $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 3e^{2t} \quad (*)$

$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 1 \quad (**)$

令 $y_1 = a\cancel{t}e^{2t}$ 代入 $(*)$ $\Rightarrow a = 3.$ $y_1 = 3te^{2t}$

$y_2 = \frac{1}{2}$

∴ 特解 $y_0 = 3te^{2t} + \frac{1}{2}$

3° 通解：$y = C_1e^t + C_2e^{2t} + 3te^{2t} + \frac{1}{2}$

$$= C_1x + C_2x^2 + 3x^2\ln x + \frac{1}{2}$$